

A note on actions of some monoids*

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Abstract

Smooth actions of the multiplicative monoid (\mathbb{R}, \cdot) of real numbers on manifolds lead to an alternative, and for some reasons simpler, definitions of a vector bundle, a double vector bundle and related structures like a graded bundle (Grabowski and Rotkiewicz (2011) [GR11]). For these reasons it is natural to study smooth actions of certain monoids closely related with the monoid (\mathbb{R}, \cdot) . Namely, we discuss geometric structures naturally related with: smooth and holomorphic actions of the monoid of multiplicative complex numbers, smooth actions of the monoid of second jets of punctured maps $(\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, smooth actions of the monoid of real 2 by 2 matrices and smooth actions of the multiplicative reals on a supermanifold. In particular cases we recover the notions of a holomorphic vector bundle, a complex vector bundle and a non-negatively graded manifold.

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1 Introduction

Motivation Our main motivation to undertake this study were the results of Grabowski and Rotkiewicz [GR09, GR11] concerning action of the multiplicative monoid of real numbers (\mathbb{R}, \cdot) on smooth manifolds. In the first of the cited papers the authors effectively characterized these smooth actions of (\mathbb{R}, \cdot) on a manifold M which come from homotheties of a vector bundle structure on M . In particular, it turned out that the addition on a vector bundle is completely determined by the multiplication by reals (yet the smoothness of this multiplication at $0 \in \mathbb{R}$ is essential). This, in turn, allowed for a simplified and very elegant treatment of double and multiple vector bundles.

These considerations were further generalized in [GR11]. The main result of that paper (here we recall it as Theorem 2.9) is an equivalence, in the categorical sense, between smooth actions of (\mathbb{R}, \cdot) on manifolds (*homogeneity structures* in the language of [GR11]) and *graded bundles*. Graded bundles (introduced for the first time in [GR11]) can be viewed as a natural generalization of vector bundles. In short, they are locally trivial fibered bundles with fibers possessing a structure of a *graded space*, i.e. a manifold diffeomorphic to \mathbb{R}^n with a distinguished class of global coordinates with positive integer

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weights assigned. In a special case when these weights are all equal to 1, a graded space becomes a standard vector space and a graded bundle – a vector bundle.

Surprisingly, graded bundles gained much more attention in supergeometry, where they are called N -manifolds. One of the reasons is that various important objects in mathematical physics can be seen as N -manifolds equipped with an odd homological vector field. For example, a Lie algebroid is a pair (E, X) where E is an N -manifold of degree 1, thus an anti-vector bundle, and X is a homological vector field on E of weight 1. A much deeper result relates Courant algebroids and N -manifolds of degree 2 [Roy02].

Goals It is natural to ask about possible extensions of the results of [GR09, GR11] discussed above. There are two obvious directions of studies:

- (Q1) What are geometric structures naturally related with smooth monoid actions on manifolds for monoids \mathcal{G} other than (\mathbb{R}, \cdot) ?
- (Q2) How to characterize smooth actions of the multiplicative reals (\mathbb{R}, \cdot) on supermanifolds?

In this paper we provide answers to the above problems. Of course it is hopeless to discuss (Q1) for an arbitrary monoid \mathcal{G} . Therefore we concentrate our attention on several special cases, all being natural generalizations of the monoid (\mathbb{R}, \cdot) of the multiplicative reals:

- $\mathcal{G} = (\mathbb{C}, \cdot)$ is the multiplicative monoid of complex numbers;
- $\mathcal{G} = \mathcal{G}_2$ is the monoid of the 2nd-jets of punctured maps $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ (note that (\mathbb{R}, \cdot) can be viewed as the monoid of the 1st-jets of such maps);
- $\mathcal{G} = M_2(\mathbb{R})$ is the monoid of 2 by 2 real matrices.

Observe that all these examples contain (\mathbb{R}, \cdot) as a submonoid. Therefore, by the results of [GR11], every manifold with a smooth \mathcal{G} -action will be canonically a graded bundle. This fact will be often of crucial importance in our analysis.

Main results Below we list the most important results of this paper regarding problem (Q1):

- For holomorphic actions of $\mathcal{G} = (\mathbb{C}, \cdot)$ we proved Theorem 3.9, a direct analog of Theorem 2.9. Such actions (*holomorphic homogeneity structures* – see Definition 3.8) are equivalent (in the categorical sense) to *holomorphic graded bundles* (see Definition 3.5) – a natural extension of the notion of a (real) graded bundle to the holomorphic setting.
- Theorem 3.18 is another analog of Theorem 2.9. It characterizes *complex graded bundles* (defined analogously to the graded bundles in the real case – see Definition 3.5) in terms of *complex homogeneity structures* (i.e., smooth actions of (\mathbb{C}, \cdot) – see Definition 3.8). It turns out that complex graded bundles are equivalent (in the categorical sense) to a special class of *nice* complex homogeneity structures, i.e. smooth (\mathbb{C}, \cdot) -actions in which the imaginary part is in a natural sense compatible with the action of $(\mathbb{R}, \cdot) \subset (\mathbb{C}, \cdot)$ – cf. Definition 3.16.
- \mathcal{G}_2 -actions on smooth manifolds are the main topic of Section 4. Since \mathcal{G}_2 is non-Abelian we have to distinguish between left and right actions of \mathcal{G}_2 . As already mentioned, since $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$, any manifold with \mathcal{G}_2 -action is naturally a (real) graded bundle. A crucial observation is that \mathcal{G}_2 contains a group of additive reals $(\mathbb{R}, +)$ as a submonoid. This fact allows to relate with every smooth right (resp., left) \mathcal{G}_2 -action a canonical complete vector field (note that a smooth action of $(\mathbb{R}, +)$ is a flow) of weight -1 (resp., $+1$) with respect to the above-mentioned graded bundle structure (Lemma 4.1).

Unfortunately, the characterization of manifolds with a smooth \mathcal{G}_2 -action as graded bundles equipped with a weight ± 1 vector field is not complete. In general, such a data allows only to define the action of the group of invertible elements $\mathcal{G}_2^{\text{inv}} \subset \mathcal{G}_2$ on the considered manifold (Lemma 4.4), still leaving open the problem of extending such an action to the whole \mathcal{G}_2 in a smooth way. For right \mathcal{G}_2 -actions this question can be locally answered for each particular case. In Lemma 4.5 we formulate such a result for graded bundles of degree at most 3. The case of a left \mathcal{G}_2 -action is much more difficult and we were able to provide an answer (in an elegant algebraic way) only for the case of graded bundles of degree one (i.e., vector bundles) in Lemma 4.6.

- As a natural application of our results about \mathcal{G}_2 -actions we were able to obtain a characterization of the smooth actions of the monoid \mathcal{G} of 2 by 2 real matrices in Lemma 4.7. This is due to the fact that \mathcal{G}_2 can be naturally embedded into \mathcal{G} . In the considered case the action of \mathcal{G} on a manifold provides it with a double graded bundle structure together with a pair of vector fields X and Y of bi-weights, respectively, $(-1, 1)$ and $(1, -1)$ with respect to the bi-graded structure. Moreover, the commutator $[X, Y]$ is related to the double graded structure on the manifold. Unfortunately, this characterization suffers the same problems as the one for \mathcal{G}_2 -actions: not every structure of such type comes from a \mathcal{G} -action.

Problem (Q2) is addressed in Section 5 where we prove, in Theorem 5.8, that supermanifolds equipped with a smooth action of the monoid (\mathbb{R}, \cdot) are *graded bundles in the category of supermanifold* in the sense of Definition 5.2 (the latter notion differs from the notion of an N -manifold given in [Roy02]: the parity of local coordinates needs not to be equal to their weights modulo two). We are aware that this result should be known to the experts. In [Sev05] Severa states (without a proof) that “an N -manifold (shorthand for ‘non-negatively graded manifold’) is a supermanifold with an action of the multiplicative semigroup (\mathbb{R}, \cdot) such that -1 acts as the parity operator”, which is a statement slightly weaker than our result. Also recently we found a proof of a result similar to Theorem 5.8 in [BGG15], Remark 2.2. Nevertheless, a rigorous proof of Theorem 5.8 seems to be missing in the literature (a version from [BGG15] is just a sketch). Therefore we decided to provide it in this paper. It is worth to stress that our proof was obtained completely independently to the one from [BGG15] and, unlike the latter, does not refer to the proof of Theorem 2.9.

Literature Despite a vast literature on Lie theory for semi-groups (see e.g., [HHL89, HH84]) we could not find anything that deals with smooth actions of the monoid (\mathbb{R}, \cdot) or its natural extensions. This can be caused by the fact that the monoid (\mathbb{R}, \cdot) is not embeddable to any group.

Organization of the paper In Section 2 we briefly recall the main results of [GR11], introducing (real) graded spaces, graded bundles, homogeneity structures, as well as the related notions and constructions. We also state Theorem 2.9 providing a categorical equivalence between graded bundles and homogeneity structures (i.e., smooth (\mathbb{R}, \cdot) -actions). Later in this section we introduce the monoid \mathcal{G}_2 and discuss its basic properties.

Section 3 is devoted to the study of (\mathbb{C}, \cdot) -actions. Basing on analogous notions from Section 2 we define complex graded spaces, complex and holomorphic graded bundles, as well as complex and holomorphic homogeneity structures. Later we prove Theorems 3.9 and 3.18 (these were already discussed above) providing effective characterizations of holomorphic and complex graded bundles, respectively, in terms of (\mathbb{C}, \cdot) -actions.

The content of Sections 4 and 5 was discussed in detail while presenting the main results of this paper.

2 Preliminaries

Graded spaces We shall begin by introducing, after [GR11], the notion of a (real) graded space. Intuitively, a graded space is a manifold diffeomorphic to \mathbb{R}^n and equipped with an atlas of global graded

coordinate systems. That is, we choose coordinate functions with certain positive integers (weights) assigned to them and consider transition functions respecting these weights (they have to be polynomial in graded coordinates). Thus a graded space can be understood as a natural generalization of the notion of a vector space. Indeed, on a vector space we can choose an atlas of global linear (weight one) coordinate systems. Clearly, every passage between two such systems is realized by a weight preserving (that is, linear) map. Below we provide a rigorous definition of a graded space.

Definition 2.1. Let $\mathbf{d} = (d_1, \dots, d_k)$ be a sequence of non-negative integers, let I be a set of cardinality $|\mathbf{d}| := d_1 + \dots + d_k$, and let $I \ni \alpha \mapsto w^\alpha \in \mathbb{Z}_+$ be a map such that $d_i = \#\{\alpha \in I : w^\alpha = i\}$ for each $1 \leq i \leq k$.

A *graded space of rank \mathbf{d}* is a smooth manifold W diffeomorphic to $\mathbb{R}^{|\mathbf{d}|}$ and equipped with an equivalence class of graded coordinates. By definition, a *system of graded coordinates* on W is a global coordinate system $(y^\alpha)_{\alpha \in I} : W \xrightarrow{\sim} \mathbb{R}^{|\mathbf{d}|}$ with *weight* w^α assigned to each function y^α , $\alpha \in I$. To indicate the presence of weights we shall sometimes write $y_{w^\alpha}^\alpha$ instead of y^α and $w(\alpha)$ to denote the weight of y^α .

Two systems of graded coordinates, $(y_{w^\alpha}^\alpha)$ and $(\underline{y}_{w^\alpha}^\alpha)$ are *equivalent* if there exist constants $c_{\alpha_1 \dots \alpha_j}^\alpha \in \mathbb{R}$, defined for indices such that $\underline{w}^\alpha = w^{\alpha_1} + \dots + w^{\alpha_j}$, satisfying

$$(1) \quad \underline{y}_{\underline{w}^\alpha}^\alpha = \sum_{\substack{j=1,2,\dots \\ \underline{w}^\alpha = w^{\alpha_1} + \dots + w^{\alpha_j}}} c_{\alpha_1 \dots \alpha_j}^\alpha y_{w^{\alpha_1}}^{\alpha_1} \dots y_{w^{\alpha_j}}^{\alpha_j}.$$

The highest coordinate weight (i.e., the highest number i such that $d_i \neq 0$) is called the *degree* of a graded space W .

By a *morphism between graded spaces* W_1 and W_2 we understand a smooth map $\Phi : W_1 \rightarrow W_2$ which in some (and thus any) graded coordinates writes as a polynomial homogeneous in weights w^α .

Example 2.2. Consider a graded space $W = \mathbb{R}^{(2,1)}$ with coordinates (x_1, x_2, y) of weights 1, 1 and 2, respectively. A map $\Phi(x_1, x_2, y) = (3x_2, x_1 + 2x_2, y + x_1x_2 + 5(x_2)^2)$ is an automorphism of W .

Observe that any graded space W induces an action $h^W : \mathbb{R} \times W \rightarrow W$ of the multiplicative monoid (\mathbb{R}, \cdot) defined by

$$(2) \quad h^W(t, (y_w^\alpha)) = (t^w \cdot y_w^\alpha).$$

Indeed, it is straightforward to check that the formula for h^W does not depend on the choice of graded coordinates (y_w^α) in a given equivalence class. We shall call h^W the action by *homotheties* of W . We will also use notation $h_t^W(\cdot)$ instead of $h^W(t, \cdot)$. The multiplicativity of h^W reads as $h_s^W \circ h_t^W = h_{ts}^W$ for every $t, s \in \mathbb{R}$.

Obviously a morphism $\Phi : W_1 \rightarrow W_2$ between two graded spaces intertwines the actions h^{W_1} and h^{W_2} , that is

$$h_t^{W_2}(\Phi(v)) = \Phi(h_t^{W_1}(v))$$

for every $v \in W_1$ and every $t \in \mathbb{R}$. For degree 1 graded spaces we recover the notion of a linear map [GR09].

Example 2.3. The graded spaces $W_1 = \mathbb{R}^{(1,0)}$ and $W_2 = \mathbb{R}^{(0,1)}$ are different although their underlying manifolds are the same. Indeed, there is no diffeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(tx) = t^2 f(x)$ for every $t, x \in \mathbb{R}$, that is one intertwining the associated actions of \mathbb{R} (cf. Lemma 2.5).

Using the above construction it is natural to introduce the following:

Definition 2.4. A function $\phi : W \rightarrow \mathbb{R}$ defined on a graded space W is called *homogeneous of weight w* if for every $v \in W$ and every $t \in \mathbb{R}$

$$(3) \quad \phi(h^W(t, v)) = t^w \phi(v) .$$

In a similar manner one can associate weights to other geometrical objects on W . For example a smooth vector field X on W is called *homogeneous of weight w* if for every $v \in W$ and every $t > 0$

$$(4) \quad (h_t)_* X(v) = t^{-w} X(h_t(v)) .$$

We see that the coordinate functions y_w^α are functions of weight w and the field $\partial_{y_w^\alpha}$ is of weight $-w$ in the sense of the above definition. In fact it can be proved that

Lemma 2.5 ([GR11]). *Any homogeneous function on a graded space W is a polynomial in the coordinate functions y_w^α homogeneous in weights w .*

Graded bundles and homogeneity structures Since, as indicated above, a graded space can be seen as a generalization of the notion of a vector space, it is natural to define *graded bundles* per analogy to vector bundles. A graded bundle is just a fiber-bundle with the typical fiber being a graded space and with transition maps respecting the graded space structure on fibers.

Definition 2.6. A *graded bundle of rank \mathbf{d}* is a smooth fiber bundle $\tau : E \rightarrow M$ over a real smooth manifold M with the typical fiber $\mathbb{R}^{\mathbf{d}}$ considered as a graded space of rank \mathbf{d} . Equivalently, τ admits local trivializations $\psi_U : \tau^{-1}(U) \rightarrow U \times \mathbb{R}^{\mathbf{d}}$ such that transition maps $g_{UU'}(q) := \psi_{U'} \circ \psi_U^{-1}|_{\{q\} \times \mathbb{R}^{\mathbf{d}}} : \mathbb{R}^{\mathbf{d}} \rightarrow \mathbb{R}^{\mathbf{d}}$ are isomorphism of graded spaces smoothly depending on $q \in U \cap U'$. By a *degree* of a graded bundle we shall understand the degree of the typical fiber $\mathbb{R}^{\mathbf{d}}$.

A *morphism of graded bundles* is defined as a fiber-bundle morphism being a graded space morphism on fibers. Clearly, graded bundles together with their morphisms form a *category*.

Example 2.7. A canonical example of a graded bundle is provided by the concept of a *higher tangent bundle*. Let M be a smooth manifold and let $\gamma, \delta : (-\varepsilon, \varepsilon) \rightarrow M$ be two smooth curves on M . We say that γ and δ have the same *k th-jet at 0* if, for every smooth function $\phi : M \rightarrow \mathbb{R}$, the difference $\phi \circ \gamma - \phi \circ \delta : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ vanishes at 0 up to order k . Equivalently, in any local coordinate system on M the Taylor expansions of γ and δ agree at 0 up to order k . The *k th-jet of γ at 0* shall be denoted by $\mathbf{t}^k \gamma(0)$. As a set the *k th-order tangent bundle* $T^k M$ consists of all *k th-jets of curves on M* . It is naturally a bundle over M with the projection $\mathbf{t}^k \gamma(0) \mapsto \gamma(0)$. It also has a natural structure of a smooth manifold and a graded bundle of rank $\underbrace{(m, m, \dots, m)}_k$ with $m = \dim M$. Indeed, given a local coordinate system

(x^i) with $i = 1, \dots, m$ on M we define the so-called *adapted coordinate system* $(x^{i,(\alpha)})$ on $T^k M$ with $i = 1, \dots, m, \alpha = 0, 1, \dots, k$ via the formula

$$x^i(\gamma(t)) = x^{i,(0)}(\mathbf{t}^k \gamma(0)) + t \cdot x^{i,(1)}(\mathbf{t}^k \gamma(0)) + \dots + \frac{t^k}{k!} x^{i,(k)}(\mathbf{t}^k \gamma(0)) + o(t^k) .$$

That is, $x^{i,(\alpha)}$ at $\mathbf{t}^k \gamma(0)$ is the α th-coefficient of the Taylor expansion of $x^i(\gamma(t))$. We can assign weight α to the coordinate $x^{i,(\alpha)}$. It is easy to check that a smooth change of local coordinates on M induces a change of the adapted coordinates on $T^k M$ which respects this grading.

Note that every graded bundle $\tau : E \rightarrow M$ induces a smooth action $h^E : \mathbb{R} \times E \rightarrow E$ of the multiplicative monoid (\mathbb{R}, \cdot) defined fiber-wise by the canonical actions h^V given by (2) with $V = \tau^{-1}(p)$ for $p \in M$. We shall refer to this as to the action by *homotheties of E* .

Clearly, $M = h_0^E(E)$ and any graded bundle morphisms $\Phi : E_1 \rightarrow E_2$ intertwine the actions h^{E_1} and h^{E_2} , i.e.,

$$\Phi(h_t^{E_1}(e)) = h_t^{E_2}(\Phi(e)),$$

for every $e \in E_1$ and every $t \in \mathbb{R}$.

The above construction justifies the following:

Definition 2.8. A *homogeneity structure* on a manifold E is a smooth action of the multiplicative monoid (\mathbb{R}, \cdot)

$$h : \mathbb{R} \times E \longrightarrow E.$$

A *morphism* of two homogeneity structures (E_1, h^1) and (E_2, h^2) is a smooth map $\Phi : E_1 \rightarrow E_2$ intertwining the actions h^1 and h^2 , i.e.,

$$\Phi(h_t^1(e)) = h_t^2(\Phi(e)),$$

for every $e \in E_1$ and every $t \in \mathbb{R}$. Clearly, homogeneity structures with their morphisms form a *category*.

In the context of homogeneity structures we can also speak about *homogeneous functions* and *homogeneous vector fields*. They are defined analogously to the notions in Definition 2.4.

As we already observed graded bundles are naturally homogeneity structures. The main result of [GR11] states that the opposite is also true: there is an equivalence between the category of graded bundles and the category of homogeneity structures (when restricted to connected manifolds).

Theorem 2.9 ([GR11]). *The category of (connected) graded bundles is equivalent to the category of (connected) homogeneity structures. At the level of objects this equivalence is provided by the following two constructions:*

- With every graded bundle $\tau : E \rightarrow M$ one can associate the homogeneity structure (E, h^E) , where h^E is the action by homotheties of E .
- Given a homogeneity structure (M, h) , the map $h_0 : M \rightarrow M_0 := h_0(M)$ provides M with a canonical structure of a graded bundle such that h is the related action by homotheties.

And at the level of morphism:

- Every graded bundle morphism $\Phi : E_1 \rightarrow E_2$ is a morphism of the related homogeneity structures (E_1, h^{E_1}) and (E_2, h^{E_2}) .
- Every homogeneity structure morphism $\Phi : (E_1, h^1) \rightarrow (E_2, h^2)$ is a morphism of graded bundles $h_0^1 : E_1 \rightarrow h_0^1(M)$ and $h_0^2 : E_2 \rightarrow h_0^2(M)$.

Let us comment briefly on the proof. The passage from graded bundles to homogeneity structures is obtained by considering the natural action by homotheties discussed above. The crucial (and difficult) part of the proof is to show that for every homogeneity structure (M, h) , the manifold M has a graded bundle structure over $h_0(M)$ compatible with the action h . The main idea is to associate to every point $p \in M$ the k -th-jet at $t = 0$ (for k big enough) of the curve $t \mapsto h_t(p)$. In this way we obtain an embedding $M \hookrightarrow T^k M$, and the graded bundle structure on M can now be naturally induced from the canonical graded bundle structure on $T^k M$ (cf. Ex. 2.7 and the proof of Theorem 3.9).

The assumption of connectedness has just a technical character: we want to prevent a situation when the fibers of a graded bundle over different base components have different ranks.

The weight vector field, the core and the natural affine fibration Let us end the discussion of graded bundles by introducing several constructions naturally associated with this notion.

Observe first that the homogeneity structure $h : \mathbb{R} \times M \rightarrow M$ provides M with a natural, globally-defined, action of the additive group $(\mathbb{R}, +)$ by the formula $(t, p) \mapsto h_{e^t}(p)$. Clearly such an action is a flow of some (complete) vector field.

Definition 2.10. A (complete) vector field Δ_M on M associated with the flow $(t, p) \mapsto h_{e^t}(p)$ is called the *weight vector field* of M . Alternatively $\Delta_M(p) = \frac{d}{dt}|_{t=1} h_t(p)$.

It is easy to show that, in local graded coordinates $(y_{w^\alpha}^\alpha)$ on M (such coordinates exist since M is a graded bundle by Theorem 2.9), the weight vector field reads as

$$\Delta_M = \sum_{\alpha} w^\alpha y_{w^\alpha}^\alpha \partial_{y_{w^\alpha}^\alpha} .$$

Actually, specifying a weight vector field is equivalent to defining the homogeneity structure on M . The passage from the weight vector field to the action of (\mathbb{R}, \cdot) is given by $h_{e^t} := \exp(t \cdot \Delta_M)$.

Remark 2.11. The assignment $\phi \mapsto w\phi$ where $\phi : M \rightarrow \mathbb{R}$ is a homogeneous function of weight w can be extended to a derivation in the algebra of smooth functions on M , thus a vector field. Clearly, it coincides with the weight vector field Δ_M what justifies the name for Δ_M . Besides, the notion of a weight vector field can be used to study the weights of certain geometrical objects defined on M (cf. Definitions 2.4 and 2.8). This should be clear, since the homogeneity structure used to define the weights can be obtained by integrating the weight vector field.

For example, X is a weight w vector field on M if and only if

$$[\Delta_M, X] = w \cdot X .$$

Using the results of Lemma 2.5, it is easy to see that, in local graded coordinates $(y_{w^\alpha}^\alpha)$, such a field has to be of the form

$$X = \sum_{\alpha} P_{\alpha} \cdot \partial_{y_{w^\alpha}^\alpha} ,$$

where P_{α} is a homogeneous function of weight $w + w^\alpha$ for each index α . Thus X , regarded as a derivation, takes a function of weight w' to a function of weight $w + w'$.

A graded bundle $\tau : E^k \rightarrow M$ of degree k ($k \geq 1$) is fibrated by submanifolds defined (invariantly) by fixing values of all coordinates of weight less or equal j ($0 \leq j \leq k$) in a given graded coordinate system. The quotient space is a graded bundle of degree j equipped with an atlas inherited from the atlas of E^k in an obvious way. The obtained bundles will be denoted by $\tau^j : E^j \rightarrow M$. They can be put together into the following sequence called *the tower of affine bundle projections associated with E^k* :

$$(5) \quad E^k \xrightarrow{\tau_{k-1}^k} E^{k-1} \xrightarrow{\tau_{k-2}^{k-1}} E^{k-2} \xrightarrow{\tau_{k-3}^{k-2}} \dots \xrightarrow{\tau_1^2} E^1 \xrightarrow{\tau^1} M .$$

Define (invariantly) a submanifold $\widehat{E^k} \subset E^k$ of a graded bundle $\tau : E \rightarrow M$ of rank (d_1, \dots, d_k) by setting to zero all fiber coordinates of degree less than k . It is a graded subbundle of rank $(0, \dots, 0, d_k)$ but we shall consider it as a vector bundle with homotheties $(t, (z_k^\alpha)) \mapsto (t \cdot z_k^\alpha)$ and call it the *core* of E^k . It is worth to note that a morphism of graded bundles respects the associated towers of affine bundle projections and induces a vector bundle morphism on the core bundles.

Example 2.12. The core of $T^k M$ is TM , while τ_{j-1}^j in the tower of affine projections associated with $T^k M$ is just the natural projection to lower-order jets $T^j M \rightarrow T^{j-1} M$.

The monoid \mathcal{G}_k We shall end this introductory part by introducing \mathcal{G}_2 , the monoid of the 2nd-jets of punctured maps $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$. Actually it is a special case of

$$(6) \quad \mathcal{G}_k := \{[\phi]_k \mid \phi : \mathbb{R} \rightarrow \mathbb{R}, \phi(0) = 0\},$$

the monoid of the k th-jets of punctured maps $\phi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$. Here $[\phi]_k$ denotes an equivalence class of the relation

$$\phi \sim_k \psi \quad \text{if and only if} \quad \phi^{(j)}(0) = \psi^{(j)}(0) \quad \text{for every } j = 1, 2, \dots, k.$$

The natural multiplication on \mathcal{G}_k is induced by the composition of maps

$$[\phi]_k \cdot [\psi]_k := [\phi \circ \psi]_k,$$

for every $\phi, \psi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$.

Thus, a class $[\phi]_k$ is fully determined by the coefficients of the Taylor expansion of ϕ at 0 up to order k and, consequently, \mathcal{G}_k can be seen as a set of polynomials of degree less or equal k vanishing at 0 equipped with a natural multiplication defined by composing polynomials and then truncating terms of order greater than k :

$$\mathcal{G}_k \simeq \{a_1 t + \frac{1}{2}a_2 t^2 + \dots + \frac{1}{k!}a_k t^k + o(t^k) : a_1, \dots, a_k \in \mathbb{R}\} \simeq \mathbb{R}^k.$$

Remark 2.13. Note that $\mathcal{G}_1 \simeq (\mathbb{R}, \cdot)$ is just the monoid of multiplicative reals, while the multiplication in $\mathcal{G}_2 \simeq \mathbb{R}^2$ is given by

$$(a, b)(A, B) = (aA, aB + bA^2).$$

Obviously,

$$(\mathbb{R}, \cdot) \simeq \{[\phi]_k : \phi(t) = at, a \in \mathbb{R}\}$$

is a submonoid of \mathcal{G}_k for every k .

Consider the set of algebra endomorphisms $\text{End}(\mathbb{D}^k)$ of the Weil algebra $\mathbb{D}^k = \mathbb{R}[\varepsilon] / \langle \varepsilon^{k+1} \rangle$. Every such an endomorphism is uniquely determined by its value on the generator ε , i.e., a map of the form

$$(7) \quad \varepsilon \longmapsto a_1 \varepsilon + \frac{1}{2}a_2 \varepsilon^2 + \dots + \frac{1}{k!}a_k \varepsilon^k.$$

It is an automorphism if $a_1 \neq 0$. Thus we may identify $\text{End}(\mathbb{D}^k)$ with \mathcal{G}_k , taking into account that the multiplication obtained by composing two endomorphisms of the form (7) is opposite to the product in \mathcal{G}_k based on the identification (6), i.e.

$$\text{End}(\mathbb{D}^k)^{\text{op}} \simeq \mathcal{G}_k.$$

Left and right monoid actions Let \mathcal{G} be an arbitrary monoid. By a *left \mathcal{G} -action* on a manifold M we understand a map $\mathcal{G} \times M \rightarrow M$ denoted by $(g, p) \mapsto g.p$ such that $h.(g.p) = (h \cdot g).p$ for every $g, h \in \mathcal{G}$ and $p \in M$. Here $h \cdot g$ denotes the multiplication in \mathcal{G} . *Right \mathcal{G} -actions* $M \times \mathcal{G} \rightarrow M$ are defined analogously. Note that if the monoid multiplication is Abelian (as is for example in the case of the multiplicative reals (\mathbb{R}, \cdot) and the multiplicative complex numbers (\mathbb{C}, \cdot)), then every left action is automatically a right action and vice versa.

Note that any left \mathcal{G} -action $(g, p) \mapsto g.p$, gives rise to a right \mathcal{G}^{op} -action on the same manifold M given by the formula $p.g = g.p$. However, unlike the case of groups actions, in general, there is no canonical correspondence between left and right \mathcal{G} -actions. For example, that is the case if $\mathcal{G} = \mathcal{G}_k$ for $k \geq 2$, since the monoids \mathcal{G}_k and $\mathcal{G}_k^{\text{op}}$ are not isomorphic.

All monoids considered in our paper will be smooth, i.e., we restrict our attention to monoids \mathcal{G} which are smooth manifolds and such that the multiplication $\cdot : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is a smooth map. We shall study smooth actions, of these monoids on manifolds, i.e. the actions $\mathcal{G} \times M \rightarrow M$ (or $M \times \mathcal{G} \rightarrow M$) which are smooth maps.

We present now two canonical examples of left and right \mathcal{G}_k -actions.

Example 2.14. The natural composition of k th-jets

$$[\gamma]_k \circ [\phi]_k \mapsto [\gamma \circ \phi]_k,$$

for $\phi : \mathbb{R} \rightarrow \mathbb{R}$, $\gamma : \mathbb{R} \rightarrow M$ where $\phi(0) = 0$ defines a right \mathcal{G}_k -action on the manifold $T^k M$.

Example 2.15. Following Tulczyjew's notation [Tul], the higher cotangent space to a manifold M at a point $p \in M$, denoted by $T_p^{k*} M$, consists of k th-jets at $p \in M$ of functions $f : M \rightarrow \mathbb{R}$ such that $f(p) = 0$. The higher cotangent bundle $T^{k*} M$ is a vector bundle whose fibers are $T_p^{k*} M$ for $p \in M$. The natural composition of k th-jets

$$[\phi]_k \circ [(f, p)]_k \mapsto [(\phi \circ f, p)]_k,$$

where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is as before, and $f : M \rightarrow \mathbb{R}$ is such that $f(p) = 0$ defines a left \mathcal{G}_k -action on $T^{k*} M$.

3 On actions of the monoid of complex numbers

In this section we study actions of the multiplicative monoid (\mathbb{C}, \cdot) on manifolds. We begin by recalling some basic construction from complex analysis, including the bundle of holomorphic jets, and by introducing the notions of a complex graded space, per analogy to Definition 2.1 in the real case. Later, again using the analogy with Section 2, we define complex and holomorphic graded bundles and homogeneity structures. The main results of this section are contained in Theorems 3.9 and 3.18. The first states that there is an equivalence between the categories of holomorphic homogeneity structures (holomorphic action of the monoid (\mathbb{C}, \cdot)) and the category of holomorphic graded bundles. This result and its proof is a clear analog of Theorem 2.9, which deals with the smooth actions of the monoid (\mathbb{R}, \cdot) . In Theorem 3.18 we establish a similar equivalence between smooth actions of (\mathbb{C}, \cdot) and complex graded bundles. However, for this equivalence to hold we need to input additional conditions (concerning, roughly speaking the compatibility between the real and the complex parts of the action) on the action of (\mathbb{C}, \cdot) in this case. We will call such actions nice complex homogeneity structures (see Definition 3.16).

Throughout this section N will be a complex manifold. We shall write $N_{\mathbb{R}}$ for the smooth manifold associated with N , thus $\dim_{\mathbb{R}} N_{\mathbb{R}} = 2 \cdot \dim_{\mathbb{C}} N$. Let $\mathcal{J} : TN_{\mathbb{R}} \rightarrow TN_{\mathbb{R}}$ be the integrable almost complex structure defined by N .

Holomorphic jet bundles Following [GG80] we shall define $J^k N$, the space of k th-jets of holomorphic curves on N , and recall its basic properties.

Consider a point $q \in N$, and let $\Delta_R \subset \mathbb{C}$ denote a disc of radius $0 < R \leq \infty$ centered at 0. Let

$$\gamma : \Delta_R \rightarrow N$$

be a holomorphic curve such that $\gamma(0) = q$. In a local holomorphic coordinate system (z^j) around $q \in N$ the curve γ is given by a convergent series

$$\gamma^j(\xi) := z^j(\gamma(\xi)) = a_0^j + a_1^j \xi + a_2^j \xi^2 + \dots, \quad |\xi| < r$$

where the coefficient a_l^j equals $\frac{1}{l!} \frac{d^l \gamma^j}{d\xi^l}(0)$ and $r \leq R$ is a positive number. We say that curves $\gamma : \Delta_R \rightarrow N$, and $\tilde{\gamma} : \Delta_{\tilde{R}} \rightarrow N$ osculate to order k if $\gamma(0) = \tilde{\gamma}(0)$ and $\frac{d^l \gamma^j}{d\xi^l}(0) = \frac{d^l \tilde{\gamma}^j}{d\tilde{\xi}^l}(0)$ for any j and

$l = 1, 2, \dots, k$. This property does not depend on the choice of a holomorphic coordinate system around q . The equivalence class of γ will be denoted by $\mathbf{j}^k \gamma(0)$ and called the *kth-jet of a holomorphic curve γ at $q = \gamma(0)$* , while the set of all such *kth-jets* at a given point q will be denoted by $\mathbf{J}_q^k N$. The totality

$$\mathbf{J}^k N = \bigcup_{q \in M} \mathbf{J}_q^k N$$

turns out to be a holomorphic (yet, in general, not linear) bundle over N . Indeed, the local coordinate system (z^j) for N gives rise to a local *adapted coordinate system* $(z^{j,(\alpha)})_{0 \leq \alpha \leq k}$ for $\mathbf{J}^k N$ where $z^{j,(\alpha)}(\mathbf{j}^k \gamma(0)) = \frac{\mathbf{d}^l \gamma^j}{\mathbf{d} \xi^l}(0)$. It is easy to see that transition functions between two such adapted coordinate systems are holomorphic. The bundle $\mathbf{J}^k N$ is called the *bundle of holomorphic kth jets of N* .

We note also that a complex curve $\gamma : \Delta_R \rightarrow N$ lifts naturally to a (holomorphic) curve $\mathbf{j}^k \gamma : \Delta_R \rightarrow \mathbf{J}^k N$. Moreover, a holomorphic map $\Phi : N_1 \rightarrow N_2$ induces a (holomorphic) map $\mathbf{J}^k \Phi : \mathbf{J}^k N_1 \rightarrow \mathbf{J}^k N_2$ defined analogously as in the category of smooth manifolds, i.e., $\mathbf{J}^k \Phi(\mathbf{j}^k \gamma) := \mathbf{j}^k(\Phi \circ \gamma)$. This construction makes \mathbf{J}^k a functor from the category of complex manifolds into the category of holomorphic bundles.

Finally, let us comment that we do not work with any fixed radius R of the disc Δ_R , being the domain of the holomorphic curve γ . Instead, by letting R be an arbitrary positive number, we work with germs of holomorphic curves. This allows to avoid technical problems related with the notion of holomorphicity. For example, according to Liouville's theorem, any holomorphic function $\phi : \Delta_R \rightarrow \mathbb{C}$ satisfies

$$|\phi^{(j)}(0)| \leq \frac{j!}{R^j} \sup_{\xi \in \Delta_R} |\phi(\xi)|.$$

It follows that, in general for a fixed radius R , coefficients a_i^j cannot be arbitrary. In particular, it may happen that there are no holomorphic curves $\gamma : \Delta_\infty \rightarrow M$ except for constant ones, or that for a fixed value $R < \infty$ there are no holomorphic curves $\gamma : \Delta_r \rightarrow M$ such that the derivatives $\frac{\mathbf{d}^l \gamma^j}{\mathbf{d} \xi^l}(0)$ have given earlier prescribed values (see for the concept of hyperbolicity and Kobayashi metric [Kob70, CW04]).

It is a well-known fact from analytic function theory, that a holomorphic map $\gamma : \Delta_R \rightarrow N$ is uniquely determined by its restriction to the real line $\mathbb{R} \subset \mathbb{C}$. Therefore it should not be surprising that the holomorphic jet bundle $\mathbf{J}^k N$ can be canonically identified with the higher tangent bundle $\mathbf{T}^k N_{\mathbb{R}}$. The latter is naturally equipped with an almost complex structure $\mathcal{J}^k : \mathbf{T} \mathbf{T}^k N_{\mathbb{R}} \rightarrow \mathbf{T} \mathbf{T}^k N_{\mathbb{R}}$ induced from \mathcal{J} , the almost complex structure on $N_{\mathbb{R}}$. Namely, $\mathcal{J}^k := \kappa_k \circ \mathbf{T}^k \mathcal{J} \circ \kappa_k^{-1}$, where $\kappa_k : \mathbf{T}^k \mathbf{T} N_{\mathbb{R}} \rightarrow \mathbf{T} \mathbf{T}^k N_{\mathbb{R}}$ is the canonical flip. In local adapted coordinates $(x^{j,(\alpha)}, y^{j,(\alpha)})$ on $\mathbf{T}^k N_{\mathbb{R}}$ such that $z^j = x^j + \sqrt{-1}y^j$ we have

$$\mathcal{J}^k \left(\frac{\partial}{\partial x^{j,(\alpha)}} \right) = \frac{\partial}{\partial y^{j,(\alpha)}}, \quad \mathcal{J}^k \left(\frac{\partial}{\partial y^{j,(\alpha)}} \right) = -\frac{\partial}{\partial x^{j,(\alpha)}}.$$

Thus the local functions $z_\alpha^j := x^{j,(\alpha)} + \sqrt{-1}y^{j,(\alpha)}$ form a system of local complex coordinates on $\mathbf{T}^k N_{\mathbb{R}}$, and hence we may treat $\mathbf{T}^k N_{\mathbb{R}}$ as a complex manifold. We identify $\mathbf{J}^k N$ with $\mathbf{T}^k N_{\mathbb{R}}$ as complex manifolds, by means of the map $\mathbf{j}^k \gamma \mapsto \mathbf{t}^k \gamma|_{\mathbb{R}}$, where $\gamma|_{\mathbb{R}}$ is the restriction of γ to the real line $\mathbb{R} \subset \mathbb{C}$. In local coordinates this identification looks rather trivially: $(z^{j,(\alpha)}) \mapsto (z_\alpha^j)$. This construction is clearly functorial, i.e., for any holomorphic map $\Phi : N_1 \rightarrow N_2$ the following diagram commutes:

$$\begin{array}{ccc} \mathbf{J}^k N_1 & \xrightarrow{\mathbf{J}^k \Phi} & \mathbf{J}^k N_2 \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbf{T}^k(N_1)_{\mathbb{R}} & \xrightarrow{\mathbf{T}^k \Phi} & \mathbf{T}^k(N_2)_{\mathbb{R}}. \end{array}$$

For $k = 1$ there is another canonical identification of the real tangent bundle $\mathbf{T} N_{\mathbb{R}}$ with the, so-called, *holomorphic tangent bundle* of N . Consider, namely, the complexification $\mathbf{T}^{\mathbb{C}} N := \mathbf{T} N_{\mathbb{R}} \otimes \mathbb{C}$

and extend \mathcal{J} to a \mathbb{C} -linear endomorphism $\mathcal{J}^{\mathbb{C}}$ of $T^{\mathbb{C}}N$. The $(+i)$ - and $(-i)$ -eigenspaces of $\mathcal{J}^{\mathbb{C}}$ define the canonical decomposition

$$T^{\mathbb{C}}N = T'N \oplus T''N$$

of $T^{\mathbb{C}}N$ into the direct sum of complex subbundles $T'N$, $T''N$ called, respectively, *holomorphic* and *anti-holomorphic tangent bundles* of N . It is easy to see, that the composition $TN_{\mathbb{R}} \subset T^{\mathbb{C}}N \rightarrow T'N$ gives a complex bundle isomorphism $TN_{\mathbb{R}} \simeq T'N$. Let $\Phi : (N_1)_{\mathbb{R}} \rightarrow (N_2)_{\mathbb{R}}$ be a smooth map and denote by $T^{\mathbb{C}}\Phi : T^{\mathbb{C}}N_1 \rightarrow T^{\mathbb{C}}N_2$ the \mathbb{C} -linear extension of $T\Phi$. It is well known that Φ is holomorphic if and only if $T\Phi$ is \mathbb{C} -linear. The latter is equivalent to $T^{\mathbb{C}}\Phi(T'N_1) \subset T'N_2$. In such a case we denote $T'\Phi := T^{\mathbb{C}}\Phi|_{T'N_1} : T'N_1 \rightarrow T'N_2$.

Thus, under the canonical identifications discussed above, all three constructions $J^1\Phi$, $T\Phi$ and $T'\Phi$ coincide (although the functor T is applicable to a wider class of maps than J^1 and T').

In what follows, given a smooth map $\phi : N \rightarrow \mathbb{C}$, the real differential of ϕ at point $q \in N$ is denoted by $d_q\phi : TN_{\mathbb{R}} \rightarrow \mathbb{C}$. If ϕ happens to be holomorphic, then $d_q\phi$ is \mathbb{C} -linear.

Holomorphic and complex graded bundles A notion of a graded space has its obvious complex counterpart. It is a generalization of a complex vector space. We basically rewrite the definitions from Section 2 in the holomorphic context.

Definition 3.1. Let $\mathbf{d} = (d_1, \dots, d_k)$ be a sequence of non-negative integers, let I be a set of cardinality $|\mathbf{d}| := d_1 + \dots + d_k$, and let $I \ni \alpha \mapsto w^\alpha \in \mathbb{Z}_+$, be a map such that $d_i = \#\{\alpha \in I : w^\alpha = i\}$ for each $1 \leq i \leq k$.

A *complex graded space of rank \mathbf{d}* is a complex manifold V biholomorphic with $\mathbb{C}^{|\mathbf{d}|}$ and equipped with an equivalence class of complex graded coordinates. By definition, a *system of complex graded coordinates* on V is a global complex coordinate system $(z^\alpha)_{\alpha \in I} : V \xrightarrow{\sim} \mathbb{C}^{|\mathbf{d}|}$ with *weight* w^α assigned to each function z^α , $\alpha \in I$. To indicate the presence of weights we shall sometimes write $z_{w^\alpha}^\alpha$ instead of z^α .

Two systems of complex graded coordinates, $(z_{w^\alpha}^\alpha)$ and $(\underline{z}_{\underline{w}^\alpha}^\alpha)$ are *equivalent* if there exist constants $c_{\alpha_1 \dots \alpha_j}^\alpha \in \mathbb{C}$, defined for indices such that $\underline{w}^\alpha = w^{\alpha_1} + \dots + w^{\alpha_j}$, satisfying

$$(8) \quad \underline{z}_{\underline{w}^\alpha}^\alpha = \sum_{\substack{j=1,2,\dots \\ \underline{w}^\alpha = w^{\alpha_1} + \dots + w^{\alpha_j}}} c_{\alpha_1 \dots \alpha_j}^\alpha z_{w^{\alpha_1}}^{\alpha_1} \dots z_{w^{\alpha_j}}^{\alpha_j}.$$

The highest coordinate weight (i.e., the highest number i such that $d_i \neq 0$) is called the *degree* of a complex graded space V .

By a *morphism between complex graded spaces* V_1 and V_2 we understand a holomorphic map $\Phi : V_1 \rightarrow V_2$ which in some (and thus any) complex graded coordinates writes as a polynomial homogeneous in weights w^α .

We remark that weights (w^α) which are assigned to coordinates on V are a part of the structure of the graded space V . Note that the set of functions $(\underline{z}_{\underline{w}^\alpha}^\alpha)$ defined by (8) defines a biholomorphic map if and only if the matrices $(c_{\beta}^\alpha)_j$ with fixed weights $w^\alpha = w^\beta = j$ are non-singular for $j = 1, 2, \dots, k$.

Remark 3.2. A complex graded space of rank $\mathbf{d} = (k)$ is just a complex vector space of dimension k . Indeed, there is no possibility to define a smooth map (a hypothetical addition of vectors in \mathbb{C}^k) $+': \mathbb{C}^k \times \mathbb{C}^k \rightarrow \mathbb{C}^k$ different from the standard addition of vectors in \mathbb{C}^k and such that \mathbb{C}^k equipped with the standard multiplication by complex numbers and addition $+'$ would satisfy all axioms of a complex vector space. This follows immediately from an analogous statement for a real vector space: $+'$ would define an alternative addition on $\mathbb{R}^{2n} \approx \mathbb{C}^n$, which is impossible.

Analogously to the real case, any complex graded space induces an action $h^V : \mathbb{C} \times V \rightarrow V$ of the multiplicative monoid (\mathbb{C}, \cdot) defined in complex graded coordinates (z_w^α) by

$$h^V(\xi, (z_w^\alpha)) = (\xi^w \cdot z_w^\alpha).$$

Here $\xi \in \mathbb{C}$. We shall call h^V the action by *homotheties* of V . Instead of $h^V(\xi, \cdot)$ we shall also write $h_\xi^V(\cdot)$.

Definition 3.3. A smooth function $f : V \rightarrow \mathbb{C}$ defined on a graded space V is called *complex homogeneous of weight w* if

$$(9) \quad f(h^V(\xi, v)) = \xi^w f(v)$$

for any $v \in V$ and any $\xi \in \mathbb{C}$.

We see that the coordinate functions z_w^α are of weight w in the sense of above definition.

Lemma 3.4. Any complex homogeneous function on a complex graded space V is a polynomial in the coordinate functions z_w^α homogeneous in weights w .

Proof. Let $\phi : V \rightarrow \mathbb{C}$ be a complex homogeneous function of weight w . Note that V can be canonically treated as a (real) graded space (of rank $2d$, where d is the rank of V), simply by assigning weights w^α to real coordinates x^α, y^α such that $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$ are complex graded coordinates of weight w^α . Let us denote this graded space by $V_{\mathbb{R}}$. Clearly, the real and imaginary parts of $\Re\phi, \Im\phi : V_{\mathbb{R}} \rightarrow \mathbb{R}$ are of weight w , in the sense of Definition 2.4, with respect to this graded structure. Thus, in view of Lemma 2.5, $\Re\phi$ and $\Im\phi$ are real polynomials (homogeneous of weight w) in x^α and y^α . We conclude that ϕ is a polynomial in z^α and \bar{z}^α with complex coefficients, homogeneous of weight w with respect to the non-standard gradation on $\mathbb{C}[z^\alpha, \bar{z}^\alpha]$ in which the weight of z^α and \bar{z}^α is w^α .

To end the proof it amounts to show that ϕ does not depend on the conjugate variables \bar{z}^α . Under an additional assumption that ϕ is holomorphic, this is immediate. For future purposes we would like, however, to assume only the smoothness of ϕ . The argument will be inductive with respect to the weight w . Cases $w = 0$ and $w = 1$ are trivial. For a general w let us fix an index α , and denote $\phi = \phi(z^\alpha, z^\beta, \bar{z}^\gamma)$ where $\beta \neq \alpha$. Denote by ϕ_α the derivative of ϕ with respect to z^α . Using the homogeneity of ϕ we easily get that for every $\xi \in \mathbb{C}$

$$\begin{aligned} \phi_\alpha(\xi^{w^\alpha} z^\alpha, \xi^{w^\beta} z^\beta, \bar{\xi}^{w^\gamma} \bar{z}^\gamma) &= \lim_{|h| \rightarrow 0} \frac{\phi(\xi^{w^\alpha} z^\alpha + h, \xi^{w^\beta} z^\beta, \bar{\xi}^{w^\gamma} \bar{z}^\gamma) - \phi(\xi^{w^\alpha} z^\alpha, \xi^{w^\beta} z^\beta, \bar{\xi}^{w^\gamma} \bar{z}^\gamma)}{h} \\ &= \lim_{|h'| \rightarrow 0} \frac{\phi(\xi^{w^\alpha} (z^\alpha + h'), \xi^{w^\beta} z^\beta, \bar{\xi}^{w^\gamma} \bar{z}^\gamma) - \phi(\xi^{w^\alpha} z^\alpha, \xi^{w^\beta} z^\beta, \bar{\xi}^{w^\gamma} \bar{z}^\gamma)}{\xi^{w^\alpha} \cdot h'} \\ &= \lim_{|h'| \rightarrow 0} \frac{\xi^w \cdot \phi(z^\alpha + h', z^\beta, \bar{z}^\gamma) - \xi^w \cdot \phi(z^\alpha, z^\beta, \bar{z}^\gamma)}{\xi^{w^\alpha} \cdot h'} \\ &= \xi^{w-w^\alpha} \phi_\alpha(\xi^{w^\alpha} z^\alpha, \xi^{w^\beta} z^\beta, \bar{\xi}^{w^\gamma} \bar{z}^\gamma). \end{aligned}$$

In other words, ϕ_α is homogeneous of weight $w - w^\alpha < w$ and thus, by the inductive assumption, a homogeneous polynomial of weight $w - w^\alpha$ in variables (z^α, z^β) . We conclude that $\phi = \psi + \eta$, where ψ is a homogeneous polynomial of weight w in variables (z^α, z^β) and η is a homogeneous polynomial of weight w in variables z^β and \bar{z}^γ where $\beta \neq \alpha$. Repeating the above reasoning several times for other indices and the polynomial η we will show that $\phi = \phi' + \eta'$, where ϕ' is a homogeneous polynomial of weight w in variables z^γ and η' is a homogeneous polynomial of weight w in variables \bar{z}^γ . In such a case η' should be a complex homogeneous function of weight w as a difference of two complex homogeneous functions ϕ and ψ' , both of weight w . However, since η' is a homogeneous polynomial in \bar{z}^γ we have

$$\eta'(\bar{\theta}^{w^\gamma} \bar{z}^\gamma) = \bar{\theta}^w \cdot \eta'(\bar{z}^\gamma),$$

hence the only possibility for η' to be complex homogeneous of weight w is $\eta' \equiv 0$. This ends the proof. \square

Analogously to the real case, we can define a complex graded bundle as a smooth fiber bundle with the typical fiber being a complex graded space. In case that the base possesses a complex manifold structure, and that the local trivializations are glued by holomorphic functions (in particular, the total space of the bundle is a complex manifold itself) we speak about holomorphic graded bundles.

Definition 3.5. A complex graded bundle of rank \mathbf{d} is a smooth fiber bundle $\tau : E \rightarrow M$ over a real smooth manifold M with the typical fiber $\mathbb{C}^{\mathbf{d}}$ considered as a complex graded space of rank \mathbf{d} . Equivalently, τ admits local trivializations $\phi_U : \tau^{-1}(U) \rightarrow U \times \mathbb{C}^{\mathbf{d}}$ such that transition functions $g_{UU'}(q) := \phi_{U'} \circ \phi_U^{-1}|_{\{q\} \times \mathbb{C}^{\mathbf{d}}} : \mathbb{C}^{\mathbf{d}} \rightarrow \mathbb{C}^{\mathbf{d}}$ are isomorphism of complex graded spaces smoothly depending on $q \in U \cap U'$. If M and E are complex manifolds and $g_{UU'}$ are holomorphic functions of q , then τ is called a *holomorphic graded bundle*.

By a *degree* of a complex (holomorphic) graded bundle we shall understand the degree of its typical fiber $\mathbb{C}^{\mathbf{d}}$.

A *morphism of complex graded bundles* is defined as a fiber-bundle morphism being a complex graded-space morphism on fibers. A *morphism of holomorphic graded bundles* is a holomorphic map between the total spaces of the considered holomorphic graded bundles being simultaneously a morphism of complex graded bundles. Clearly, complex (holomorphic) graded bundles together with their morphisms form a *category*.

Remark 3.6. Note that a complex graded bundle $\tau : E \rightarrow M$ in which E and M are complex manifolds and the projection τ is a holomorphic map needs not to be a holomorphic graded bundle. We also need to assume that the action h is also holomorphic, i.e. the complex structure on each fiber is actually the restriction of the holomorphic structure of E . To see this consider a complex rank 1 vector bundle $C \subset E := \mathbb{C}^* \times \mathbb{C}^2$ ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$) given in natural holomorphic coordinates $(x; y^1, y^2)$ on E by the equation

$$C = \{(x; y^1, y^2) : xy^1 + \bar{x}y^2 = 0, x \in \mathbb{C}^*\}.$$

We shall construct a degree 2 complex (but not holomorphic) graded bundle structure on E . Set $Y^1 := xy^1 + \bar{x}y^2$ and $Y^2 := -xy^1 + \bar{x}y^2$. We may take $(x; Y^1, Y^2)$ as a global coordinate system on E and assign weights 1, 2 to Y^1, Y^2 , respectively, to define a complex graded bundle structure on the fibration $\tau : E \rightarrow \mathbb{C}^*$. Then C coincides with the core of E , which in every holomorphic graded bundle should be a complex submanifold of the total space. However, C is not a complex submanifold of E , and thus it is impossible to find a homogeneous holomorphic atlas on E . The associated action by homotheties $h : \mathbb{C} \times E \rightarrow E$ reads as

$$h(\xi, (x; y^1, y^2)) = (x; \frac{1}{2}(\xi + \xi^2)y^1 + \frac{\bar{x}}{2x}(\xi - \xi^2)y^2, \frac{x}{\bar{x}}(\xi - \xi^2)y^1 + \frac{1}{2}(\xi + \xi^2)y^2).$$

This action is smooth but not holomorphic, hence it induces a complex but not holomorphic graded bundle structure according to the forthcoming Theorems 3.9 and 3.18.

In what follows, to avoid possible confusions, we will use notation $\tau : E \rightarrow M$ for complex and smooth graded bundles and $\tau : F \rightarrow N$ for holomorphic graded bundles.

Example 3.7. The holomorphic jet bundle $J^k N$ is a canonical example of a holomorphic (and thus also complex) graded bundle. This fact is justified analogously to the real case (see Ex. 2.7).

Finally, we can rewrite Definition 2.8 in the complex context.

Definition 3.8. A *complex* (respectively, *holomorphic*) *homogeneity structure* on a smooth (resp., complex) manifold M is a smooth (resp., holomorphic) action of the multiplicative monoid (\mathbb{C}, \cdot)

$$h : \mathbb{C} \times M \longrightarrow M.$$

A *morphism* of two complex (resp., holomorphic) homogeneity structures (M_1, h^1) and (M_2, h^2) is a smooth (resp., holomorphic) map $\Phi : M_1 \rightarrow M_2$ intertwining the actions h^1 and h^2 , i.e.,

$$\Phi(h_\xi^1(p)) = h_\xi^2(\Phi(p)),$$

for every $p \in M_1$ and every $\xi \in \mathbb{C}$. Clearly, complex (resp., holomorphic) homogeneity structures with their morphisms form a *category*.

It is clear, that with every complex (holomorphic) graded bundle $\tau : E \rightarrow M$ one can associate a natural complex (holomorphic) homogeneity structure $h^E : \mathbb{C} \times E \rightarrow E$ defined fiber-wise by the canonical (\mathbb{C}, \cdot) actions h^V where $V = \tau^{-1}(p)$ for every $p \in M$. We shall call it the action by *homotheties* of E . In the remaining part of this section we shall study the relations between the notions of a homogeneity structure and a graded bundle in the complex and holomorphic settings. Our goal is to obtain analogs of Theorem 2.9 in these two situations.

A holomorphic action of the monoid of complex numbers In the holomorphic setting the results of Theorem 2.9 have their direct analog.

Theorem 3.9. *The categories of (connected) holomorphic graded bundles and (connected) holomorphic homogeneity structures are equivalent. At the level of objects this equivalence is provided by the following two constructions*

- With every holomorphic graded bundle $\tau : F \rightarrow N$ one can associate the holomorphic homogeneity structure (F, h^F) , where h^F is the action by the homotheties of F .
- Given a holomorphic homogeneity structure (N, h) , the map $h_0 : N \rightarrow N_0 := h_0(N)$ provides N with a canonical structure of a holomorphic graded bundle such that h is the related action by homotheties.

At the level of morphisms: every morphism of holomorphic graded bundles is a morphism of the related holomorphic homogeneity structures and, conversely, every morphism of holomorphic homogeneity structures respects the related canonical holomorphic graded bundle structures.

To prove the above theorem we will need two technical results.

Lemma 3.10. *Let N be a connected complex manifold and let $\Phi : N \rightarrow N$ be a holomorphic map satisfying $\Phi \circ \Phi = \Phi$. Then the image $N_0 := \Phi(N)$ is a complex submanifold of N .*

Proof. An analogous result for smooth manifolds is given in Theorem 1.13 in [KMS93]. Its proof can be almost directly rewritten in the complex setting. Namely, from the proof given in [KMS93] we know that there is an open neighborhood U of N_0 in N such that the tangent map $T_p\Phi : T_pN_{\mathbb{R}} \rightarrow T_{\Phi(p)}N_{\mathbb{R}}$ has a constant rank while p varies in U . Therefore, due to the identification $T_pN_{\mathbb{R}} \approx T'_pN$, the map $T'_p\Phi : T'_pN \rightarrow T'_{\Phi(p)}N$ also has a constant rank.

Now take any $q \in N_0$, so $\Phi(q) = q$. From the constant rank theorem for complex manifolds (see e.g., [Gau14] or [KK83]) we can find two charts (O, v) and $(\underline{O}, \underline{v})$ on N , both centered at q , such that $\underline{v} \circ \Phi \circ v^{-1}$ is a projection of the form $(z^1, \dots, z^n) \mapsto (z^1, \dots, z^{\underline{n}}, 0, \dots, 0)$, where $\underline{n} \leq n$ is the rank of $T'_q\Phi$. We conclude that $O \cap N_0$ is a complex submanifold of N of dimension \underline{n} . Since $q \in N_0$ was arbitrary and N_0 is connected, the assertion follows. \square

Lemma 3.11. *Let V be a complex graded space and $V' \subset V$ a complex submanifold invariant with respect to the action of the homotheties of V , i.e., $h_\xi^V(V') \subset V'$ for any $\xi \in \mathbb{C}$. Then V' is a complex graded subspace of V .*

Proof. Observe first that $0 = h_0^V(V') = h_0^V(V)$ lies in V' . Denote by $(z_w^\alpha)_{\alpha \in I}$ a system of complex graded coordinates on V . Since V' is a complex submanifold, we may choose a subset $I' \subset I$ of cardinality $\dim_{\mathbb{C}} V'$ such that the differentials

$$(10) \quad d_q z_w^\alpha|_{T_q' V'}, \quad \alpha \in I',$$

are linearly independent (over \mathbb{C}) at $q = 0$. In consequence, the restrictions $(z_w^\alpha|_{V'})_{\alpha \in I'}$ form a coordinate system for V' around 0. The idea is to show that these functions form a global graded coordinate system on V' .

Note that V has an important property that it can be recovered from an arbitrary open neighborhood of 0 by the action of h^V . Since $V' \subset V$ is h^V -invariant it also has this property. As has been already showed, the differentials (10) are linearly independent for $q \in U \cap V'$ where U is a small neighborhood of 0 in V . Using the equality

$$\langle d_{h(\xi, q)} f_w^i, (Th_\xi) v_q \rangle = \xi^w \langle d_q f_w^i, v_q \rangle$$

where $f_w^i : V \rightarrow \mathbb{C}$ is a function of weight w and $v_q \in T_q V$, and the property that V' is generated by h^V from any neighborhood of 0, we conclude that the differentials (10) are linearly independent for any $q \in V'$. Thus $(z_w^\alpha|_{V'})_{\alpha \in I'}$ is a global system of graded coordinates for V' . This ends the proof. \square

Corollary 3.12. *Let $\tau' : E' \rightarrow M'$ be a complex graded subbundle of a holomorphic graded bundle $\tau : F \rightarrow M$ such that E' is a complex submanifold of F . Then E' is a holomorphic graded subbundle.*

Proof. First of all the base $M' := M \cap E'$ of E' is a complex submanifold of M . To prove this we apply Lemma 3.10 with $\Phi = \tau|_{E'} : E' \rightarrow E'$.

Being a holomorphic subbundle is a local property: if any point $q \in M'$ has an open neighborhood $U' \subset M'$ such that $E'|_{U'}$ is a holomorphic subbundle, then E' is itself a holomorphic subbundle of F . Indeed, we know that transition maps of E' are holomorphic since E' is a holomorphic submanifold. Moreover, by Lemma 3.4, these maps are also polynomial on fibers.

Thus take $q \in M'$ and denote graded fiber coordinates of $\tau : F \rightarrow M$ by $(z_w^\alpha)_{\alpha \in I}$. They are holomorphic functions defined on $F|_U$ for some open subset $U \subset M$, $q \in U$. Let (x^a) be coordinates on $U' \subset M'$ around q . Take a subset $I' \subset I$ such that functions $(x^a, z_w^\alpha)_{\alpha \in I'}$ form a coordinate system for E around q . Then the differentials $(\alpha \in I')$

$$d_{\tilde{q}} z_w^\alpha|_{T_q' E'}, \quad d_{\tilde{q}} x^a|_{T_q' E'}$$

are still linearly independent for \tilde{q} in some open neighborhood \tilde{U} of q in F , possibly smaller than U . It follows from the proof of Lemma 3.11 that $(z_w^\alpha)_{\alpha \in I'}$ form a global coordinate system on each of the fibers of E' over $q' \in U'' := \tilde{U} \cap U'$. Thus $(x^a, z_w^\alpha)_{\alpha \in I'}$ is a graded coordinate system for the subbundle $E'|_{U''}$ consisting of holomorphic functions turning it into a holomorphic graded bundle. This finishes the proof. \square

Now we are ready to prove Theorem 3.9.

Proof of Theorem 3.9: The crucial step is to show, that a holomorphic action $h : \mathbb{C} \times N \rightarrow N$ of the multiplicative monoid (\mathbb{C}, \cdot) on a connected complex manifold N , determines the structure of a holomorphic graded bundle on $h_0 : N \rightarrow N_0 := h_0(N)$.

Clearly, since h is holomorphic, so is h_0 . What is more, this map satisfies $h_0 \circ h_0 = h_0$ and thus, by Lemma 3.10, $N_0 = h_0(N)$ is a complex submanifold of N . Now notice that the restriction of h to $\mathbb{R} \times N$ gives an action

$$h^{\mathbb{R}} : \mathbb{R} \times N \rightarrow N$$

of the monoid (\mathbb{R}, \cdot) . In view of Theorem 2.9, $h_0 : N \rightarrow N_0$ is a (real) graded bundle (cf. the proof of Lemma 3.4), say, of degree k .

We shall now follow the ideas from the proof of Theorem 2.9 provided in [GR11]. The crucial step is to embed N into the holomorphic jet bundle $J^k N$ as a holomorphic graded subbundle. Recall (cf. Ex. 3.7) that $J^k N$ has a canonical holomorphic graded bundle structure. Consider a map

$$\phi^{\mathbb{C}} : N \rightarrow J^k N, \quad q \mapsto \mathbf{j}_{\xi=0}^k h(\xi, q), \quad \xi \in \mathbb{C},$$

sending each point $q \in N$ to the k th-holomorphic jet at $\xi = 0$ of the holomorphic curve $\mathbb{C} \ni \xi \mapsto h(\xi, q) \in N$. Clearly, $\phi^{\mathbb{C}}$ is a holomorphic map as a lift of a holomorphic curve to the holomorphic jet bundle $J^k N$. The composition of $\phi^{\mathbb{C}}$ with the canonical isomorphism $J^k N \simeq T^k N_{\mathbb{R}}$ gives the map

$$\phi^{\mathbb{R}} : N_{\mathbb{R}} \rightarrow T^k N_{\mathbb{R}}, \quad q \mapsto \mathbf{t}_{t=0}^k h(t, q), \quad t \in \mathbb{R}.$$

As indicated in the proof of Theorem 4.1 of [GR11], $\phi^{\mathbb{R}}$ is a topological embedding naturally related with the real homogeneity structure $h^{\mathbb{R}}$. Thus also $\phi^{\mathbb{C}}$ is a topological embedding. Therefore, since $\phi^{\mathbb{C}}$ is also holomorphic, the image $\tilde{N} := \phi^{\mathbb{C}}(N) \subset J^k N$ is a complex submanifold, biholomorphic with N . Let us denote by $\tilde{h} : \mathbb{C} \times \tilde{N} \rightarrow \tilde{N}$ the corresponding action on \tilde{N} induced from h by means of $\phi^{\mathbb{C}}$. Since $\phi^{\mathbb{C}}$ intertwines the action h and the canonical action by homotheties

$$h^{J^k N} : \mathbb{C} \times J^k N \rightarrow J^k N$$

on the bundle of holomorphic k th-jets, the action \tilde{h} coincides with the restriction of $h^{J^k N}$ to $\mathbb{C} \times \tilde{N}$. Hence, \tilde{N} is a complex submanifold of $J^k N$ invariant with respect to the action $h^{J^k N}$. Using Lemma 3.11 on each fiber of $J^k N|_{N_0} \rightarrow N_0$ we conclude that \tilde{N} is a complex graded subbundle of $J^k N$. Since $\tilde{N} \approx N$ was a complex submanifold of $J^k N$, it is also a holomorphic graded subbundle of $J^k N$ due to Corollary 3.12. Thus we have constructed a canonical holomorphic graded bundle structure on N starting from a holomorphic homogeneity structure (N, h) . Clearly the action by homotheties h^N related with this graded bundle coincides with the initial action h .

The above construction, and the construction of a canonical holomorphic homogeneity structure (F, h^F) from a holomorphic graded bundle $\tau : F \rightarrow N$ are mutually inverse, providing the desired equivalence of categories at the level of objects.

To show the equivalence at the level of morphisms consider two holomorphic graded bundles $\tau_j : F_j \rightarrow N_j$, with $j = 1, 2$, and let h^j , with $j = 1, 2$, be the related homogeneity structures. Let $\Phi : F_1 \rightarrow F_2$ be a holomorphic map such that $\Phi \circ h^1 = h^2 \circ \Phi$. It is enough to show that Φ is a morphism of holomorphic graded bundles. Since Φ is holomorphic by assumption it suffices to show that on each fiber $\Phi : (F_1)_p \rightarrow (F_2)_{\Phi(p)}$ is a morphism of complex graded spaces.

Let now (z_w^α) and (z_w^α) be graded coordinates on $(F_1)_p$ and $(F_2)_{\Phi(p)}$, respectively. Note that $\Phi^* z_w^\alpha = z_w^\alpha \circ \Phi$ is a \mathbb{C} -homogeneous function on $(F_1)_p$, hence in light of Lemma 3.4 it is a homogeneous polynomial in z_w^α . Thus, indeed, Φ has a desired form. \square

On smooth actions of the monoid (\mathbb{C}, \cdot) on smooth manifolds Our goal in the last paragraph of this section is to study smooth actions of the monoid (\mathbb{C}, \cdot) on smooth manifolds, i.e., complex homogeneity structures (see Definition 3.8). Contrary to the holomorphic case, there is no equivalence between such structures and complex graded bundles. To guarantee such an equivalence we will need to make additional assumptions. Informally speaking, the real and the imaginary parts of the action of (\mathbb{C}, \cdot) should be compatible. The following examples should help to get the right intuitions.

Example 3.13. Consider $M = \mathbb{R}$ and define an action $h : \mathbb{C} \times M \rightarrow M$ by $h(\xi, y) = (|\xi|^2 y)$, where y is a standard coordinate on \mathbb{R} . Clearly this is a smooth action of the multiplicative monoid (\mathbb{C}, \cdot) , but the fibers M admit no structure of a complex graded bundle. This is clear from dimensional reasons. Indeed, the base $h_0(M)$ is just a single point $0 \in \mathbb{R}$ and thus M , as a single fiber, should admit a structure of a complex graded space. This is impossible as M is odd-dimensional.

Example 3.14. Consider $M = \mathbb{C}$ with a standard coordinate $z : M \rightarrow \mathbb{C}$ and define a smooth multiplicative action $h : \mathbb{C} \times M \rightarrow M$ by the formula $h(\xi, z) = |\xi|^2 \xi z$. We claim that h is not a homothety action related with any complex graded bundle structure on M .

Assume the contrary. The basis of M should be $h_0(M) = \{0\}$, i.e. a single point. Thus M is a complex graded space (a complex graded bundle over a single point), say, of rank d . Clearly the restriction $h|_{\mathbb{R}} : \mathbb{R} \times M \rightarrow M$ should provide M with a structure of a (real) graded space of rank $2d$. Observe that $h|_{\mathbb{R}}$ is in fact a homothety action on $\mathbb{R}^{(0,0,2)}$ and thus we should have $M = \mathbb{C}^{(0,0,1)}$. In such a case, there should exist a global complex coordinate $\tilde{z} : M \xrightarrow{\sim} \mathbb{C}$ which is homogeneous of degree 3, and so h_{ε_3} , where $\varepsilon_3 := e^{2\pi\sqrt{-1}/3}$ is the third order primitive root of 1, should be the identity on M , as there are no coordinates on M of other weights. Yet, $h(\varepsilon_3, z) = \varepsilon_3 z \neq z$, thus a contradiction.

The above examples reveal two important facts concerning a complex homogeneity structure $h : \mathbb{C} \times M \rightarrow M$. First of all, the restriction of h to (\mathbb{R}, \cdot) makes M a (real) homogeneity structure. Secondly, the action of the primitive roots of 1 on $h|_{\mathbb{R}}$ -homogeneous functions allows to distinguish complex graded bundles among all complex homogeneity structures.

Remark 3.15. Let $h : \mathbb{C} \times E^k \rightarrow E^k$ be a complex homogeneity structure such that the restriction $h|_{\mathbb{R}}$ makes $\tau = h_0 : E^k \rightarrow M_0$ a (real) graded bundle of degree k . For any $\xi \in \mathbb{C}$ the action h_ξ commutes with the homotheties $h(t, \cdot)$, $t \in \mathbb{R}$, hence $h_\xi : E^k \rightarrow E^k$ is a (real) graded bundle morphism, in view of Theorem 2.9. Therefore h induces an action of the monoid (\mathbb{C}, \cdot) on each (real) graded bundle $\tau^j : E^j \rightarrow M_0$ in the tower (5), and on each core bundle \hat{E}^j , for $j = 1, 2, \dots, k$.

These observations motivate the following

Definition 3.16. Let $h : \mathbb{C} \times E^k \rightarrow E^k$ be a complex homogeneity structure such that $\tau = h_0 : E^k \rightarrow M$ is the (real) graded bundle of degree k associated with $h|_{\mathbb{R}}$. Denote by $\varepsilon_{2j} := e^{2\pi\sqrt{-1}/(2j)}$ the $2j$ th-order primitive root of 1, and by $J_{2j} := h(\varepsilon_{2j}, \cdot) : E^k \rightarrow E^k$ the action of ε_{2j} on E^k . We say that the complex homogeneity structure h is *nice* if J_{2j} acts as minus identity on the core bundle \hat{E}^j for every $j = 1, 2, \dots, k$.

Nice complex homogeneity structures form a *full subcategory* of the category of complex homogeneity structures.

Example 3.17. It is easy to see, using local coordinates, that if $\tau : E \rightarrow M$ is a complex graded bundle, and $h^E : \mathbb{C} \times E \rightarrow E$ the related complex homogeneity structure, then h^E is nice.

It turns out that the converse is also true, that is, for nice complex homogeneity structures we can prove an analog of Theorem 3.9:

Theorem 3.18. *The categories of (connected) complex graded bundles and (connected) nice complex homogeneity structures are equivalent. At the level of objects this equivalence is provided by the following two constructions*

- With every complex graded bundle $\tau : E \rightarrow M$ one can associate a nice complex homogeneity structure (E, h^E) , where h^E is the action by homotheties of E .
- Given a nice complex homogeneity structure (M, h) , the map $h_0 : M \rightarrow M_0 := h_0(M)$ provides M with a canonical structure of a complex graded bundle such that h is the related action by homotheties.

At the level of morphisms: every morphism of complex graded bundles is a morphism of the related nice complex homogeneity structures and, conversely, every morphism of nice complex homogeneity structures respects the canonical complex graded bundle structures.

Again in the proof we shall need a few technical results. First observe that $J_2 = h_{-1}$ acts as minus identity on every vector bundle, so for $k = 1$ the condition in Definition 3.16 is trivially satisfied (i.e., a degree-one complex homogeneity structure is always nice). Thus every (\mathbb{C}, \cdot) -action whose restriction to (\mathbb{R}, \cdot) is of degree one should be a complex bundle.

Lemma 3.19. *Let $h : \mathbb{C} \times W \rightarrow W$ be a smooth action of the monoid (\mathbb{C}, \cdot) on a real vector space W , such that $h(t, v) = t v$ for every $t \in \mathbb{R}$ and $v \in W$.*

Then h induces a complex structure on W by the formula

$$(11) \quad h(a + b\sqrt{-1}, v) = a v + b h(\sqrt{-1}, v)$$

for every $a, b \in \mathbb{R}$.

Proof. Denote $J := h(\sqrt{-1}, \cdot) : W \rightarrow W$. We have $J \circ J = h_{\sqrt{-1}} \circ h_{\sqrt{-1}} = h_{-1} = -\text{id}_W$. Moreover, J is \mathbb{R} -linear, since J commutes with the homotheties $h_t : W \rightarrow W$ for any $t \in \mathbb{R}$ (see Theorem 2.4 [GR09]). Therefore, J defines a complex structure on W and the formula

$$\xi v := \Re \xi v + \Im \xi J(v),$$

where $\xi \in \mathbb{C}$ and $v \in W$, allows us to consider W as a complex vector space. Note that $\xi v = h(\xi, v)$ for $\xi \in \mathbb{R}$. To prove that this equality holds for any $\xi \in \mathbb{C}$ we shall study the restriction

$$h|_{S^1 \times W} : S^1 \times W \rightarrow W$$

which is a group action of the unit circle on a complex vector space W . Indeed, for any $\xi \in \mathbb{C}$, the action $h(\xi, \cdot) : W \rightarrow W$ is a \mathbb{C} -linear map, since it commutes with the complex structure J and the endomorphisms $h(t, \cdot)$ for $t \in \mathbb{R}$. It follows from the general theory that W splits into sub-representations $W = \bigoplus_{j=1}^n W_j$ such that for any $|\theta| = 1$ and any $v \in W_j$

$$h(\theta, v) = \theta^{k_j} v$$

where k_1, \dots, k_n are some integers. The restriction of h to each summand W_j defines an action of the monoid (\mathbb{C}, \cdot) hence, without loss of generality, we may assume that $W = W_1$ and that

$$(12) \quad h(t\theta, v) = t\theta^k v$$

for every $\theta \in S^1$, $t \in \mathbb{R}$ and $v \in W$. Note that k should be an odd integer as $J^2 = -\text{id}_W$. Equivalently, taking $\xi = t\theta$, we can denote $h(\xi, v) = \xi^k \bar{\xi}^{-k+1} v$ for every $\xi \in \mathbb{C} \setminus \{0\}$. However, the function $\xi \mapsto \xi^k \bar{\xi}^{-k+1}$, $0 \mapsto 0$, is not differentiable at $\xi = 0$ unless $k = 1$. Therefore, $h(\xi, v) = \xi v$, as was claimed. \square

Now we shall show that an analogous result holds for nice homogeneity structures of arbitrary degree.

Lemma 3.20. *Let $h : \mathbb{C} \times M \rightarrow M$ be a nice complex homogeneity structure such that the restriction $h|_{\mathbb{R}}$ makes M a (real) graded space of degree k . Then M is a complex graded space of degree k with homotheties given by h .*

Proof. Denote by W^k the (real) graded space structure on M . By W^j with $j \leq k$ denote the lower levels of the tower (5) associated with W^k . We shall proceed by induction on k . Case $k = 1$ follows immediately from Lemma 3.19.

Let now k be arbitrary. The basic idea of the proof is to define a complex graded space structure of degree $k - 1$ on W^{k-1} using the inductive assumption and to construct a complex graded space structure of rank $(0, \dots, 0, \dim_{\mathbb{C}} \widehat{W}^k)$ on the core \widehat{W}^k . Then using both structures we build complex graded coordinates on W^k .

Recall (see Remark 3.15) that, for any $\xi \in \mathbb{C}$, the map $h_\xi : W^k \rightarrow W^k$ is a (real) graded space morphism, and that h induces an action of the monoid (\mathbb{C}, \cdot) on each graded space W^j in the tower (5). Note that the induced action on W^{k-1} satisfies all assumptions of our lemma hence, by the inductive assumption, we may consider W^{k-1} as a complex graded space.

Denote by (z_w^α) complex graded coordinates on W^{k-1} and pullback them to W^k by means of the projection $\tau_{k-1}^k : W^k \rightarrow W^{k-1}$. Denote the resulting functions again with the same symbols. This should not lead to any confusion since the pullbacked function $z_w^\alpha : W^k \rightarrow \mathbb{C}$ is still of weight w , i.e.

$$(13) \quad h_\xi^*(z_w^\alpha) = \xi^w z_w^\alpha.$$

On the other hand, as any morphism of graded spaces, h_ξ can be restricted to the core \widehat{W}^k . This defines an action of (\mathbb{C}, \cdot) on \widehat{W}^k ,

$$\widehat{h} := h|_{\mathbb{C} \times \widehat{W}^k} : \mathbb{C} \times \widehat{W}^k \rightarrow \widehat{W}^k.$$

Recall that \widehat{W}^k is a real vector space with homotheties defined by

$$(14) \quad (t, v) \mapsto t * v := h|_{\mathbb{R}}(\sqrt[k]{t}, v),$$

for every $t \geq 0$ and every $v \in \widehat{W}^k$. In a (real) graded coordinate system $(\Re z_w^\alpha, \Im z_w^\alpha, p^\mu)$ on W^k , where $p^\mu : W^k \rightarrow \mathbb{R}$ are arbitrary coordinates of weight k , the map (14) reads $(t, (\widehat{p}^\mu)) \mapsto (t \widehat{p}^\mu)$, where $\widehat{p}^\mu = p^\mu|_{\widehat{W}^k}$. Hence it is a smooth map (with respect to the inherited submanifold structure) which can be extended also to the negative values of t .

Let us denote by \widehat{J}_{4k} the restriction of $J_{4k} := h(\varepsilon_{4k}, \cdot)$ to \widehat{W}^k . By assumption (h is nice), $-\text{id}_{\widehat{W}^k} = \widehat{J}_{2k} = \widehat{J}_{4k} \circ \widehat{J}_{4k}$. Therefore (cf. Lemma 3.19), \widehat{J}_{4k} defines a complex structure on the real vector space \widehat{W}^k by the formula

$$(a + b\sqrt{-1}) * v := a * v + b * \widehat{J}_{4k}(v),$$

for every $a, b \in \mathbb{R}$ and $v \in \widehat{W}^k$.

Note that the homotheties h_ξ commute also with \widehat{J}_{4k} , therefore $h_\xi|_{\widehat{W}^k}$ is a \mathbb{C} -linear endomorphism of \widehat{W}^k . By restricting to $|\xi| = 1$ we obtain a representation of the unit circle group S^1 in $\text{GL}_{\mathbb{C}}(\widehat{W}^k)$. As in the proof of Lemma 3.19, without loss of generality we may assume that there exists an integer m such that

$$\widehat{h}(\theta, v) = \theta^m * v$$

for every $|\theta| = 1$. Taking $\theta = \varepsilon_{2k}$ we see that $\varepsilon_{2k}^m = -1$, hence $m \equiv k \pmod{2k}$. It follows that $\widehat{h}(t\theta, v) = t^k \theta^m * v$, for every $t \in \mathbb{R}$ and $\theta \in S^1$. However, the function $\xi := t\theta \mapsto t^k \theta^m = \xi^k (\xi/\bar{\xi})^{m-k}$ is smooth only if $m = k$, since $m - k$ is a multiplicity of $2k$. Therefore,

$$(15) \quad h(\xi, v) = \xi^k * v,$$

for every $\xi \in \mathbb{C}$ and every $v \in \widehat{W}^k$. In other words, \widehat{W}^k is a complex graded space of rank $\mathbf{d} = (0, 0, \dots, 0, \dim_{\mathbb{C}} \widehat{W}^k)$.

Let $(\widehat{z}^\mu : \widehat{W}^k \rightarrow \mathbb{C})$ be a system of complex graded coordinates on \widehat{W}^k . We shall show that it is possible to extend each \widehat{z}^μ to a complex function $z_k^\mu : W^k \rightarrow \mathbb{C}$ in such a way that

$$(16) \quad z_k^\mu(h(\xi, v)) = \xi^k z_k^\mu(v)$$

hold for any $\xi \in \mathbb{C}$ and $v \in W^k$, i.e., z_k^μ are complex homogeneous function of weight k . First see that we can find extensions of \widehat{z}^μ which satisfy (16) for $\xi \in \mathbb{R}$. Indeed, the restriction to \widehat{W}^k of a real homogeneous weight k function $W^k \rightarrow \mathbb{R}$ can be an arbitrary linear function on \widehat{W}^k . Thus we extend the real and imaginary parts of \widehat{z}^μ separately and get \mathbb{R} -homogeneous extensions, say $\tilde{z}^\mu : W^k \rightarrow \mathbb{C}$. Now consider a function

$$(17) \quad z_k^\mu(v) := \frac{1}{2\pi} \int_{|\xi|=1} \xi^{-k} \tilde{z}^\mu(h_\xi(v)) d\lambda(\xi),$$

where $v \in W^k$ and λ is a homogeneous measure on the circle $S^1 = \{|\xi| = 1\}$ with $\lambda(S^1) = 2\pi$. Clearly, $z_k^\mu : W^k \rightarrow \mathbb{C}$ is smooth and (as $h_\xi(h_\theta(v)) = h_{\xi\theta}(v)$ for any $\theta, \xi \in \mathbb{C}$) we have

$$z_k^\mu(h_\theta(v)) = \theta^k \frac{1}{2\pi} \int_{|\xi|=1} \theta^{-k} \xi^{-k} \tilde{z}^\mu(h_{\xi\theta}(v)) d\lambda(\xi) = \theta^k z_k^\mu(v),$$

for any $|\theta| = 1$. Since \tilde{z}^μ is \mathbb{R} -homogeneous, the same is z_k^μ , hence z_k^μ is actually a complex homogeneous function, as S^1 and \mathbb{R} generate \mathbb{C} as a monoid. Moreover, for $v \in \widehat{W}^k$ equality $\tilde{z}^\mu(h_\xi(v)) = \xi^k \tilde{z}^\mu(v)$ holds, hence $z_k^\mu|_{\widehat{W}^k} = \tilde{z}^\mu|_{\widehat{W}^k}$, and so z_k^μ is indeed an extension of \widehat{z}^μ .

Lastly, the system of homogeneous functions (z_w^α, z_k^μ) defines a global diffeomorphism $W^k \xrightarrow{\sim} \mathbb{C}^{|\mathbf{d}|}$ where $\mathbf{d} = (d_1, \dots, d_k)$, $d_j = \dim_{\mathbb{C}} \widehat{W}^j$. Indeed, it is enough to point that for any $1 \leq j \leq k$, the restrictions of some of these functions (those of weight j) to the core \widehat{W}^j define a diffeomorphism $\widehat{W}^j \xrightarrow{\sim} \mathbb{C}^{d_j}$. \square

Theorem 3.18 is a simple consequence of the above result.

Proof of Theorem 3.18. We already observed (see Ex. 3.17) that a complex graded bundle structure on $\tau : E \rightarrow M$ induces a nice complex homogeneity structure (E, h^E) by the associated action of the homotheties of E .

The converse is also true. Indeed, let $h : \mathbb{C} \times M \rightarrow M$ be a nice complex homogeneity structure. Clearly $h|_{\mathbb{R}}$ is a (real) homogeneity structure on M , and thus, by Theorem 2.9, $h_0 : M \rightarrow M_0 := h_0(M)$ is a (real) graded bundle of degree, say, k . Now on each fiber of this bundle the action h defines a nice homogeneity structure. By applying Lemma 3.20 we get a complex graded space structure on each fiber of h_0 . Thus M is indeed a complex graded bundle.

The equivalence at the level of morphisms is showed analogously to the holomorphic case: it amounts to show that a smooth map between two complex graded spaces which intertwines the homothety actions is a complex graded space morphism. This is precisely the assertion of Lemma 3.4. \square

4 Actions of the monoid \mathcal{G}_2

In this section we shall study smooth actions of the monoid \mathcal{G}_2 on smooth manifolds.

The left and right actions of the monoid \mathcal{G}_2 Recall (see the end of Section 2) that \mathcal{G}_2 was introduced as the space of 2nd-jets of punctured maps $\phi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ with the multiplication induced by the composition. Under the identification of \mathcal{G}_2 with $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ it reads as (see Remark 2.13):

$$(18) \quad (a, b)(A, B) = (aA, aB + bA^2).$$

Since this multiplication is clearly non-commutative, unlike in the case of real or complex numbers, we have to distinguish between left and right actions of \mathcal{G}_2 .

The crucial observation about \mathcal{G}_2 is that it contains two submonoids:

- the multiplicative reals $(\mathbb{R}, \cdot) \simeq \{(a, 0) : a \in \mathbb{R}\}$, corresponding to the 2nd-jets of punctured maps $\phi(t) = at$ for $a \in \mathbb{R}$,
- and the additive group $(\mathbb{R}, +) \simeq \{(1, b) : b \in \mathbb{R}\}$.

Now to study right (or left) smooth actions of \mathcal{G}_2 on a smooth manifold M we use a technique similar to the one used to study (\mathbb{C}, \cdot) -actions in Section 3. We begin by considering the action of $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$ which, by Theorem 2.9, makes M a (real) graded bundle. Actually it will be more convenient to speak of the related weight vector field (see Definition 2.10) in this case. On the other hand, the action of the additive reals $(\mathbb{R}, +)$ is a flow, i.e. it is encoded by a single (complete) vector field on M . It is now crucial to understand the relation (compatibility conditions) between these two structures. This can be done by looking at the formula

$$(19) \quad (a, 0)(1, b/a) = (a, b) = (1, b/a^2)(a, 0) ,$$

which allows to decompose every element of $\mathcal{G}_2^{\text{inv}} = \mathcal{G}_2 \setminus \{(0, b) : b \neq 0\}$, the group of invertible elements of \mathcal{G}_2 , as a product of the elements of the submonoids (\mathbb{R}, \cdot) and $(\mathbb{R}, +)$. Since equation (19) describes the commutation of the two submonoids, it helps to express the compatibility conditions of the two related structures at the infinitesimal level as the following result shows. Recall the notion of a homogeneous vector field – cf. Definition 2.4 and Remark 2.11.

Lemma 4.1. *Every smooth right (respectively, left) action $H : M \times \mathcal{G}_2 \rightarrow M$ (resp., $H : \mathcal{G}_2 \times M \rightarrow M$) on a smooth manifold M provides M with:*

- a canonical graded bundle structure $\pi : M \rightarrow M_0 := H_{(0,0)}(M)$ induced by the action of the submonoid $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$,
- and a complete vector field $X \in \mathfrak{X}(M)$ of weight -1 (resp., $Y \in \mathfrak{X}(M)$ of weight $+1$) with respect to the above graded structure on M .

In other words, any \mathcal{G}_2 -action provides M with two complete vector fields: the weight vector field Δ and another vector field X (respectively, Y), such that their Lie bracket satisfies

$$(20) \quad [\Delta, X] = -X \quad (\text{resp., } [\Delta, Y] = Y) .$$

Proof. By Theorem 2.9, the homogeneity structure $h : \mathbb{R} \times M \rightarrow M$ obtained as the restriction of H to the submonoid $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$ (i.e., $h_a = H_{(a,0)}$) defines a graded bundle structure on $h_0 : M \rightarrow M_0$. Clearly, the flow of the corresponding weight vector field Δ is given by $t \mapsto H_{(e^t, 0)}$.

Consider first the case when H is a right \mathcal{G}_2 -action. Let $X \in \mathfrak{X}(M)$ be the infinitesimal generator of the action of $s \mapsto H_{(1,s)}$. In order to calculate the Lie bracket of Δ and X we will calculate the corresponding commutator of flows, i.e.

$$\begin{aligned} X^{-s} \circ \Delta^{-t} \circ X^s \circ \Delta^t &= H_{(1,-s)} \circ H_{(e^{-t}, 0)} \circ H_{(1,s)} \circ H_{(e^t, 0)} \\ &= H_{(e^t, 0)(1,s)(e^{-t}, 0)(1,-s)} \stackrel{(18)}{=} H_{(1, -ts + o(ts))} = X^{-ts + o(ts)} . \end{aligned}$$

The latter should correspond to the ts -flow of $[\Delta, X]$ and hence (20) holds.

In the case when H is a left \mathcal{G}_2 -action denote by Y the infinitesimal generator of the action $s \mapsto H_{(1,s)}$. Now $H_g \circ H_{g'} = H_{gg'}$, so the commutator $Y^{-s} \Delta^{-t} \circ Y^s \circ \Delta^t$ equals $H_{(1,-s)(e^{-t}, 0)(1,s)(e^t, 0)} = Y^{ts + o(ts)}$, hence $[\Delta, Y] = Y$. \square

Example 4.2. Consider the right \mathcal{G}_2 -action on T^2M described in Example 2.14. We shall use standard coordinates $(x^i, \dot{x}^i, \ddot{x}^i)$ on T^2M .¹ We have

$$(x^i, \dot{x}^i, \ddot{x}^i).(a, b) = (x^i, a\dot{x}^i, a^2\ddot{x}^i + b\dot{x}^i).$$

It is clear that in this case the homogeneity structure on T^2M is just the standard degree 2 homogeneity structure, the weight vector field equals $\Delta = \dot{x}^i \partial_{\dot{x}^i} + 2 \ddot{x}^i \partial_{\ddot{x}^i}$, and that the additional vector field X of weight -1 is simply $X = \dot{x}^i \partial_{\ddot{x}^i}$.

Example 4.3. Let us now focus on the left \mathcal{G}_2 -action on $T^{2*}M$ described in Example 2.15. We shall use standard coordinates (p_i, p_{ij}) on $T^{2*}M$.² We have

$$(a, b).(p_i, p_{ij}) = (ap_i, ap_{ij} + bp_i p_j),$$

so $\Delta = p_i \partial_{p_i} + p_{ij} \partial_{p_{ij}}$ and $Y = p_i p_j \partial_{p_{ij}}$ has indeed weight 1 with respect to the standard vector bundle structure on $T^{2*}M$.

Our goal now is to prove the inverse of Lemma 4.1, i.e. to characterize right (resp., left) \mathcal{G}_2 -actions in terms of a homogeneity structure and a complete vector field X of weight -1 (resp., Y of weight $+1$). Obviously, by our preliminary considerations knowing the actions of the two canonical submonoids of \mathcal{G}_2 allows to determine the action of the Lie subgroup $\mathcal{G}_2^{\text{inv}}$ of invertible elements of \mathcal{G}_2 . Yet problems with extending this action on the whole \mathcal{G}_2 may appear.

Lemma 4.4. Let $\tau : M \rightarrow M_0$ be a graded bundle (with the associated weight vector field Δ and the corresponding homogeneity structure $h : \mathbb{R} \times M \rightarrow M$) and let $X \in \mathfrak{X}(M)$ (resp., $Y \in \mathfrak{X}(M)$) be a complete vector field of weight -1 (resp., $+1$) i.e. $[\Delta, X] = -X$ (resp., $[\Delta, Y] = Y$). Then the formulas

$$(21) \quad p.(a, b) := X^{b/a}(h_a(p)) = h_a(X^{b/a^2}(p)) \quad \left(\text{resp., } (a, b).p := h_a(Y^{b/a}(p)) = Y^{b/a^2}(h_a(p)) \right)$$

define a smooth right (resp., left) action of the group of invertible elements $\mathcal{G}_2^{\text{inv}} \subset \mathcal{G}_2$. Here $t \mapsto X^t$ (resp., $t \mapsto Y^t$) denotes the flow of the vector field X (resp., Y).

Proof. We shall restrict our attention to the case of the right action. The reasoning for the case of the left action is analogous.

Note that the Lie algebra generated by vector fields Δ and X is a non-trivial two-dimensional Lie algebra, thus it is isomorphic to the Lie algebra $\text{aff}(\mathbb{R})$ of the Lie group

$$\text{Aff}(\mathbb{R}) = \left\{ \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} : c \neq 0, d \in \mathbb{R} \right\} \subset \text{GL}_2(\mathbb{R})$$

of affine transformations of \mathbb{R} . The identification $\mathcal{G}_2^{\text{inv}} \simeq \text{Aff}(\mathbb{R})$ is given by

$$(a, b) \mapsto \begin{pmatrix} 1/a & b/a^2 \\ 0 & 1 \end{pmatrix}.$$

Let $\text{Aff}_+(\mathbb{R})$ be the subgroup of orientation-preserving affine transformations of \mathbb{R} ,

$$\text{Aff}_+(\mathbb{R}) = \left\{ \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} : c > 0, d \in \mathbb{R} \right\}.$$

The subgroup $\text{Aff}_+(\mathbb{R})$ is the connected and simply-connected Lie group integrating $\text{aff}(\mathbb{R})$. Clearly under the above identification it is isomorphic to $\mathcal{G}_{2+}^{\text{inv}} = \{(a, b) \in \mathcal{G}_2 : a > 0\}$. Thus, due to Palais'

¹If $[\gamma]_2 \sim (x^i, \dot{x}^i, \ddot{x}^i)$, then $\gamma(t) = (x^i + t\dot{x}^i + \frac{1}{2}\ddot{x}^i t^2 + o(t^2))$.

²If $[(f, x)] \sim (p_i, p_{ij})$ then $f(x) = 0$ and $f(x+h) = p_i h^i + \frac{1}{2} p_{ij} h^i h^j + o(|h|^2)$, where $p_{ij} = p_{ji}$.

theorem and according to (19), formula (21) is a well-defined action of the group $\mathcal{G}_{2+}^{\text{inv}}$ on M , i.e., both formulas $p_{.1}(a, b) := X^{b/a}(h_a(p))$ and $p_{.2}(a, b) := h_a(X^{b/a^2}(p))$ coincide for $(a, b) \in \mathcal{G}_{2+}^{\text{inv}}$ and the resulting map is indeed a right action of $\mathcal{G}_{2+}^{\text{inv}}$.

Our goal now is to show that (21) is a well-defined action of the whole $\mathcal{G}_2^{\text{inv}}$. It is straightforward to check that $p_{.1}(-1, 0) = p_{.2}(-1, 0) = h_{-1}(p)$. Now let us check that the formulas for $.1$ and $.2$ coincide for elements of $\mathcal{G}_2^{\text{inv}} \setminus \mathcal{G}_{2+}^{\text{inv}}$. Indeed, observe first that by Definition 2.4, since X is homogeneous of weight -1 , for every $a \in \mathbb{R}$

$$(h_a)_* X_p = a X_{h_a(p)}.$$

Integrating the above equality we obtain the following result for flows:

$$h_a(X^t(p)) = X^{ta}(h_a(p)),$$

for every $a, t \in \mathbb{R}$. Using this result we get for $a > 0$

$$p_{.1}(-a, b) = X^{-b/a}(h_{-a}(p)) = h_{-a}(X^{b/a^2}(p)) = p_{.2}(-a, b),$$

i.e., (21) is well-defined on the whole $\mathcal{G}_2^{\text{inv}}$. To check that this is indeed an action of $\mathcal{G}_2^{\text{inv}}$ note that

$$p_{.1}(-a, b) = h_{-a}(X^{b/a^2}(p)) = h_{-1} \left[h_a(X^{b/a^2}(p)) \right] = [p_{.1}(a, b)]_{.1}(-1, 0)$$

and

$$p_{.2}(-a, b) = X^{-b/a}(h_{-a}(p)) = X^{-b/a}(h_a(h_{-1}(p))) = [p_{.2}(-1, 0)]_{.2}(a, -b).$$

In other words, the operation $p \mapsto p_{.1}(a, b)$ is compatible with the following decomposition

$$(22) \quad (-a, b) = (a, b)(-1, 0) = (-1, 0)(a, -b).$$

Now it suffices to observe that the latter formula allows to express every multiplication of two elements in $\mathcal{G}_2^{\text{inv}}$ as a composition of multiplications of elements of $\mathcal{G}_{2+}^{\text{inv}}$ and $(-1, 0)$ (in other words, $\mathcal{G}_2^{\text{inv}}$ is a semi-direct product of $\mathcal{G}_{2+}^{\text{inv}}$ and $C_2 \simeq \{(\pm 1, 0)\}$). Since formula (21) is multiplicative with respect to $\mathcal{G}_{2+}^{\text{inv}}$ and C_2 and respects (22), it is a well-defined action of the whole $\mathcal{G}_2^{\text{inv}}$. \square

We have thus shown that the infinitesimal data related with the right (resp., left) action of \mathcal{G}_2 on a smooth manifold M , i.e., a weight vector field Δ together with a complete vector field X (resp., Y) on M satisfying (20), integrates to a right (resp., left) action of the Lie group $\mathcal{G}_2^{\text{inv}}$ on M . However, there is no guarantee that this action will extend to the action of the whole $\mathcal{G}_2 \supset \mathcal{G}_2^{\text{inv}}$. This will happen if formula (21) has a well-defined and smooth extension to $a = 0$. In particular situations this condition can be checked by a direct calculation, yet no general criteria are known to us.

In the forthcoming paragraphs we shall study (local) conditions of this kind after restrict ourselves to the cases when the graded bundle (M, Δ) is of low degree. The cases of left and right actions turned out to be essentially different and so we treat them separately.

Right \mathcal{G}_2 -actions of degree at most 3 Let us now classify (locally) all possible right \mathcal{G}_2 -actions on a smooth manifold M such that the associated graded bundle structure (M, Δ) is of degree at most 3. That is, locally on M we have graded coordinates $(x^i, y_1^s, y_2^S, y_3^\sigma)$ where the lower index indicates the degree. By the results of Remark 2.11, the general formula for a vector field X of degree -1 in such a setting is

$$(23) \quad X = F^s(x) \partial_{y_1^s} + G_s^S(x) y_1^s \partial_{y_2^S} + \left(H_S^\sigma(x) y_2^S + \frac{1}{2} I_{sr}^\sigma(x) y_1^s y_1^r \right) \partial_{y_3^\sigma},$$

where $F^s, G_s^S, H_S^\sigma, I_{sr}^\sigma$ are smooth functions on the base. The following result characterizes these fields X which give rise to a right action of the monoid \mathcal{G}_2 on M :

Lemma 4.5. *Let (M, Δ) be a graded bundle of degree at most 3 and let X be a weight -1 vector field on M given locally by formula (23). Then the right action $H : M \times \mathcal{G}_2^{\text{inv}} \rightarrow M$ defined in Lemma 4.4 extends to a smooth right action of \mathcal{G}_2 on M if and only if $F^s = 0$ and $H_S^\sigma G_s^S = 0$ for every indices s and σ . Equivalently, X is a degree -1 vector field tangent to the fibration $M = M^3 \rightarrow M^1$, such that the differential weight -2 operator $X \circ X$ vanishes on all functions on M of weight less or equal 3.*

Proof. In order to find the flow $t \mapsto X^t$ we need to solve the following system of ODEs

$$\begin{aligned}\dot{x}^i &= 0 \\ \dot{y}_1^s &= F^s(x) \\ \dot{y}_2^S &= G_s^S(x) y_1^s(t) \\ \dot{y}_3^\sigma &= H_S^\sigma(x) y_2^S(t) + \frac{1}{2} I_{sr}^\sigma(x) y_1^s(t) y_1^r(t),\end{aligned}$$

which gives the following output

$$\begin{aligned}x^i(t) &= x^i(0), \\ y_1^s(t) &= y_1^s(0) + t F^s, \\ y_2^S(t) &= y_2^S(0) + t G_s^S y_1^s(0) + \frac{1}{2} t^2 G_s^S F^s, \\ y_3^\sigma(t) &= y_3^\sigma(0) + t (H_S^\sigma y_2^S(0) + \frac{1}{2} I_{sr}^\sigma y_1^s(0) y_1^r(0)) + \frac{1}{2} t^2 (H_S^\sigma G_s^S y_1^s(0) + I_{sr}^\sigma F^s y_1^r(0)) \\ &\quad + \frac{1}{6} t^3 (H_S^\sigma G_s^S F^s + I_{sr}^\sigma F^s F^r).\end{aligned}$$

Now, by Lemma 4.4, the action of $(a, b) \in \mathcal{G}_2$ on $p \in M$ should be defined as $h_a \left(X^{b/a^2}(p) \right)$, that is, it affects the coordinate y_w^α of weight w by $y_w^\alpha \mapsto a^w y_w^\alpha (b/a^2)$. Thus we have

$$\begin{aligned}H_{(a,b)}^* x^i &= x^i, \\ H_{(a,b)}^* y_1^s &= a y_1^s + \frac{b}{a} F^s, \\ H_{(a,b)}^* y_2^S &= a^2 y_2^S + b G_s^S y_1^s + \frac{1}{2} \frac{b^2}{a^2} G_s^S F^s, \\ H_{(a,b)}^* y_3^\sigma &= a^3 y_3^\sigma + ab (H_S^\sigma y_2^S + I_{sr}^\sigma y_1^s y_1^r) + \frac{1}{2} \frac{b^2}{a} (H_S^\sigma G_s^S y_1^s + 2 I_{sr}^\sigma F^s y_1^r) + \frac{1}{6} \frac{b^3}{a^3} (H_S^\sigma G_s^S F^s + I_{sr}^\sigma F^s F^r).\end{aligned}$$

Now it is clear that the action $H_{(a,b)}$ depends smoothly on (a, b) if and only if $F^s = 0$ and $H_S^\sigma G_s^S = 0$.

The vector field X is tangent to the fibration $M^3 \rightarrow M^1$ if and only if $F^s = 0$. Then the condition on the differential operator $X \circ X$ means that $0 = X(X(y_3^\sigma)) = X(X(y_1^s y_2^S))$ which simplifies to $H_S^\sigma G_s^S = 0$. \square

Note that in degree 1 (i.e., when (M, Δ) is a vector bundle) X has to be the zero vector field, hence $v.(a, b) = a \cdot v$ for any $(a, b) \in \mathcal{G}_2$ and $v \in M$.

In degree 2 the only possibility is $X = G_s^S y_1^s \partial_{y_2^S}$. In geometric terms, (G_s^S) defines a vector bundle morphism

$$\phi : F^1 \rightarrow \widehat{F^2}, \quad \phi(x, y_1^s) = (x, \widehat{y}_2^S = G_s^S(x) y_1^s),$$

covering id_{M_0} where $M = F^2 \rightarrow F^1 \rightarrow M_0$ is the tower of affine bundle projections (5) associated with (M, Δ) . Thus, there is a one-to-one correspondence between degree 2 right \mathcal{G}_2 -actions and vector bundle morphisms ϕ as above. The correspondence is defined by the formula

$$v.(a, b) = h_a(v) + b \phi(v),$$

where $v \in F^2$; h_a , for $a \in \mathbb{R}$, are homotheties of (F^2, Δ) ; and $+$: $F^2 \times_{M_0} \widehat{F^2} \rightarrow F^2$ is the canonical action of the core bundle on a graded bundle. Obviously in higher degrees finding the precise conditions for X gets more complicated (yet is still doable in finite time) and more classes of admissible weight -1 vector fields appear.

Left \mathcal{G}_2 -actions In case of left \mathcal{G}_2 -actions we meet a problem of integrating a vector field Y of weight 1. Even if the associated graded bundle (M, Δ) is a vector bundle (a graded bundle of degree 1), a vector field of weight 1 has a general form

$$\frac{1}{2} F_{ij}^k(x) y^i y^j \partial_{y^k} + F_i^a(x) y^i \partial_{x^a}$$

and, in general, is not integrable in quadratures. Therefore, the problem of classifying left \mathcal{G}_2 -action seems to be more difficult. We will solve it in the simplest case when (M, Δ) is a vector bundle.

Lemma 4.6. *Let $\tau : E \rightarrow M$ be a vector bundle. There is a one-to-one correspondence between smooth left \mathcal{G}_2 -actions on E such that the multiplicative submonoid $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$ acts by the homotheties of E and symmetric bi-linear operations $\bullet : E \times_M E \rightarrow E$ such that for any $v \in E$*

$$(24) \quad v \bullet (v \bullet v) = 0 .$$

This correspondence is given by the following formula

$$(25) \quad (a, b).v = a v + b v \bullet v ,$$

where $(a, b) \in \mathcal{G}_2$ and $v \in E$.

Proof. We shall denote the action of an element $(a, b) \in \mathcal{G}_2$ on $v \in E$ by $(a, b).v$. Observe first that we can restrict our attention to a single fiber of E . Indeed, since $\tau(v) = (0, 0).v$, we have $\tau((a, b).v) = (0, 0).(a, b).v = (0, 0).v = \tau(v)$ and thus $(a, b).v$ belongs to the same fiber of E as v does. In consequence, without any loss of generality, we may assume that E is a vector space.

By the results of Lemma 4.1, every left \mathcal{G}_2 -action on E induces a weight 1 homogeneous vector field $Y \in \mathfrak{X}(E)$. The flow of such a Y at time t corresponds to the action of an element $(1, t) \in \mathcal{G}_2$.

Choose now a basis $\{e_i\}_{i \in I}$ of E and denote by $\{y^i\}_{i \in I}$ the related linear coordinates. In this setting (cf. Remark 2.11) Y writes as

$$Y = \frac{1}{2} F_{ij}^k y^i y^j \partial_{y^k}, \quad \text{where} \quad F_{ij}^k = F_{ji}^k .$$

Let us now define the product \bullet on base elements of E by the formula

$$e_i \bullet e_j = F_{ij}^k e_k ,$$

and extend it be-linearly to an operation $\bullet : E \times_M E \rightarrow E$. In other words, $Y(v) = v \bullet v$, where we use the canonical identification of the vertical tangent vectors of TE with elements of E .

We shall now show that the action of \mathcal{G}_2 is given by formula (25). Recall that by Lemma 4.4 the action of $(a, b) \in \mathcal{G}_2^{\text{inv}}$ on $v \in E$ is given by

$$(26) \quad (a, b).v = a \cdot v(b/a) ,$$

where $t \mapsto v(t) := (1, t).v$ denotes the integral curve of Y emerging from $v(0) = v$ at $t = 0$. The question is whether the above formula extends smoothly to the whole \mathcal{G}_2 .

Note that for $t \neq 0$ we have $(t, tb) \cdot v = tv(b)$. Thus, assuming the existence of a smooth extension of (26) to the whole \mathcal{G}_2 , we have

$$\left. \frac{d}{dt} \right|_{t=0} (t, tb) \cdot v = v(b) .$$

On the other hand, by the Leibniz rule we can write

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (t, tb) \cdot v &= \left. \frac{d}{dt} \right|_{t=0} (t, 0) \cdot v + \left. \frac{d}{dt} \right|_{t=0} (0, tb) \cdot v = \left. \frac{d}{dt} \right|_{t=0} (t, 0) \cdot v + \left. \frac{d}{dt} \right|_{t=0} (tb, 0) \cdot (0, 1) \cdot v \\ &= \left. \frac{d}{dt} \right|_{t=0} t \cdot v + \left. \frac{d}{dt} \right|_{t=0} tb \cdot (0, 1) \cdot v = v + b \cdot (0, 1) \cdot v . \end{aligned}$$

We conclude that for every $b \in \mathbb{R}$

$$(27) \quad v(b) = v + b \cdot (0, 1) \cdot v ,$$

i.e., integral curves of Y are straight lines or constant curves. Differentiating the above formula with respect to b we get $(0, 1) \cdot v = Y(v) = v \bullet v$. Using this and (26) we get for $(a, b) \in \mathcal{G}_2^{\text{inv}}$

$$(a, b) \cdot v = a \cdot v(b/a) = a(v + b/a \cdot v \bullet v) = a \cdot v + b \cdot v \bullet v .$$

Clearly this formula extends smoothly to the whole \mathcal{G}_2 . We have thus proved that any smooth \mathcal{G}_2 -action on E such that $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$ acts by the homotheties of E is of the form (25).

Clearly formula (25) considered for some, a priori arbitrary, bi-linear operation \bullet defines a left \mathcal{G}_2 -action if and only if $(aA, aB + bA^2) \cdot v \stackrel{(18)}{=} (a, b)(A, B) \cdot v$ for every $v \in E$ and $(a, b), (A, B) \in \mathcal{G}_2$. It is a matter of a simple calculation to check that this requirement leads to the following condition:

$$2AbBv \bullet (v \bullet v) + bB^2(v \bullet v) \bullet (v \bullet v) = 0 .$$

Since v and a, b, A and B were arbitrary this is equivalent to $v \bullet (v \bullet v) = 0$ and $(v \bullet v) \bullet (v \bullet v) = 0$ for every $v \in E$. To end the proof it amounts to show that this latter condition is induced by the former one. Indeed, after a short calculation formula (24) considered for $v = v' + t \cdot w$ leads to the following condition

$$t[w \bullet (v' \bullet v') + 2v' \bullet (v' \bullet w)] + t^2[v' \bullet (w \bullet w) + 2w \bullet (w \bullet v')] = 0 .$$

Thus, as $t \in \mathbb{R}$ was arbitrary, $w \bullet (v' \bullet v') + 2v' \bullet (v' \bullet w) = 0$ for every $v', w \in E$. In particular, taking $w = v' \bullet v'$ and using (24) we get $(v' \bullet v') \bullet (v' \bullet v') = 0$. This ends the proof. \square

Actions of the monoid of 2 by 2 matrices Let $\mathcal{G} := M_{2 \times 2}(\mathbb{R})$ be the monoid of 2 by 2 matrices with the natural matrix multiplication. We shall end our considerations in this section by studying smooth actions of this structure on manifolds.

We have a canonical isomorphism $\mathcal{G} \simeq \mathcal{G}^{\text{op}}$ which sends a matrix to its transpose. Thus, unlike the case of \mathcal{G}_2 , left and right \mathcal{G} -actions are in one-to-one correspondence. Moreover, any \mathcal{G} -action gives rise to left and right \mathcal{G}_2 -action as there is a canonical monoid embedding $\mathcal{G}_2 \rightarrow \mathcal{G}$, $(a, b) \mapsto \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix}$. This observation allows to prove easily the following result.

Lemma 4.7. *Any \mathcal{G} -action on a manifold M gives rise to a double graded bundle (M, Δ_1, Δ_2) equipped with two complete vector fields X, Y of weights $(1, -1)$ and $(-1, 1)$ respectively, such that $[X, Y] = \Delta^1 - \Delta^2$.*

Proof. Since the homogeneity structures defined by the actions of the submonoids $G_1 = \{\text{diag}(t, 1) : t \in \mathbb{R}\}$, $G_2 = \{\text{diag}(1, t) : t \in \mathbb{R}\}$, $G_1 \simeq (\mathbb{R}, \cdot) \simeq G_2$, commute, the corresponding weight vector fields Δ_1, Δ_2 also commute and give rise to a double graded structure (M, Δ_1, Δ_2) . Define vector fields X, Y as infinitesimal actions of the subgroups $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, respectively. It is straightforward to check, as we did for \mathcal{G}_2 -actions, that $[\Delta_1, X] = -X$, $[\Delta_2, X] = X$, so X has weight $w(X) = (-1, 1)$. Similarly, $[\Delta_1, Y] = Y$, $[\Delta_2, Y] = -Y$, so $w(Y) = (1, -1)$, and moreover

$$Y^{-s} \circ X^{-t} \circ Y^s \circ X^t = \begin{pmatrix} 1 - ts & -t^2s \\ s^2t & 1 + st + o(st) \end{pmatrix}$$

so $[X, Y] = \Delta_2 - \Delta_1$ as we claimed. \square

Example 4.8. Let M be a manifold and consider the space $J^2_{(0,0)}(\mathbb{R} \times \mathbb{R}, M)$ of all 2nd-jets at $(0, 0)$ of maps $\gamma : \mathbb{R}^2 \rightarrow M$. Given local coordinates (x^j) on M , the adapted local coordinates $(x^j_{00}, x^j_{10}, x^j_{01}, x^j_{20}, x^j_{11}, x^j_{02})$ on $J^2_{(0,0)}(\mathbb{R}^2, M)$ of $[\gamma]_2$ are defined as coefficients of the Taylor expansion

$$(28) \quad \gamma(t, s) = (\gamma^j(t, s)), \quad \gamma^j(t, s) = x^j_{00} + x^j_{10}t + x^j_{01}s + x^j_{20}\frac{t^2}{2} + x^j_{11}ts + x^j_{02}\frac{s^2}{2} + o(t^2, ts, s^2).$$

The right action of $A \in \mathcal{G}$ on $[\gamma]_2 \in M$ equals $[\gamma(at + bs, ct + ds)]_2$ and reads as

$$(29) \quad (x^j_{00}, x^j_{10}, x^j_{01}, x^j_{20}, x^j_{11}, x^j_{02}) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x^j_{00}, ax^j_{10} + cx^j_{01}, bx^j_{10} + dx^j_{01}, a^2x^j_{20} + 2acx^j_{11} + c^2x^j_{02}, abx^j_{02} + (ad + bc)x^j_{11} + cdx^j_{02}, b^2x^j_{20} + 2bdx^j_{11} + d^2x^j_{02})$$

Hence the action of $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ yields a vector field $X = x^j_{10}\partial_{x^j_{01}} + x^j_{20}\partial_{x^j_{11}} + 2x^j_{11}\partial_{x^j_{02}}$ of weight $(1, -1)$.

Similarly, the action of $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ gives rise to a vector field $Y = x^j_{01}\partial_{x^j_{10}} + x^j_{02}\partial_{x^j_{11}} + 2x^j_{11}\partial_{x^j_{20}}$ of weight $(-1, 1)$. We have

$$[X, Y] = x^j_{10}\partial_{x^j_{10}} - x^j_{01}\partial_{x^j_{01}} + 2x^j_{20}\partial_{x^j_{20}} - 2x^j_{02}\partial_{x^j_{02}} = \Delta_1 - \Delta_2,$$

where $\Delta_1 = x^j_{10}\partial_{x^j_{10}} + 2x^j_{20}\partial_{x^j_{20}} + x^j_{11}\partial_{x^j_{11}}$, $\Delta_2 = x^j_{01}\partial_{x^j_{01}} + x^j_{11}\partial_{x^j_{11}} + 2x^j_{02}\partial_{x^j_{02}}$ are commuting weight vector fields. This example has a direct generalization for the case of higher order $(1, 1)$ -velocities.

5 On actions of the monoid of real numbers on supermanifolds

Super graded bundles The notions of a super vector bundle (see e.g. [BCC11]) and a graded bundle generalize naturally to the notion of a *super graded bundle*, i.e., a graded bundle in the category of supermanifolds. The latter is a *super fiber bundle* (see e.g., [BCC11]) $\pi : \mathcal{E} \rightarrow \mathcal{M}$ in which one can distinguish a class of \mathbb{N} -graded fiber coordinates so that transition functions preserve this gradation (Definition 5.2). On the other hand, super graded bundles are a particular example of non-negatively graded manifolds in the sense of Voronov [Vor02]. These are defined as supermanifolds with a privileged class of atlases in which one assigns \mathbb{N}_0 -weights to particular coordinates. Coordinates of positive weights are ‘cylindrical’ and coordinate changes are decreed to be polynomials which preserve $\mathbb{Z}_2 \times \mathbb{N}_0$ -gradation. The coordinate parity is not determined by its \mathbb{N}_0 -weight.

Our goal in this section is to prove a direct analog of Theorem 2.9 in supergeometry: (\mathbb{R}, \cdot) -actions on supermanifolds are in one-to-one correspondence with super graded bundles.

To fix the notation, given a supermanifold defined by its structure sheaf (M, \mathcal{O}_M) , we shall usually denote it shortly by \mathcal{M} . Here $M := |\mathcal{M}|$ is a topological space called the *body* of \mathcal{M} . Elements of $\mathcal{O}_M(U)$ will be called *local functions* on \mathcal{M} . For an open subset U of M let $\mathcal{J}_M(U)$ be the ideal of nilpotent elements in $\mathcal{O}_M(U)$. The quotient sheaf $\mathcal{O}_M/\mathcal{J}_M$ defines a structure of a real smooth manifold on the body $|\mathcal{M}|$. For local functions $f, g \in \mathcal{O}_M(U)$ a formula $f = g + o(\mathcal{J}_M^i)$ means that $f - g \in (\mathcal{J}_M(U))^i$.

The definition of a super graded bundle, alike its classical analog, will be given in steps. We begin by introducing the notion of a super graded space, which is, basically speaking, a superdomain $\mathbb{R}^{m|n}$ equipped with an atlas of global graded coordinates.

Definition 5.1. Let $\mathbf{d} := (\mathbf{d}_0 | \mathbf{d}_1)$, where $\mathbf{d}_\varepsilon = (d_{\varepsilon,1}, \dots, d_{\varepsilon,k})$, $\varepsilon \in \mathbb{Z}_2$, $1 \leq i \leq k$ are sequences of non-negative integers, and let $|\mathbf{d}_\varepsilon| := \sum_{i=1}^k d_{\varepsilon,i}$.

A *super graded space of rank \mathbf{d}* is a supermanifold W isomorphic³ to a superdomain $\mathbb{R}^{|\mathbf{d}_0| | |\mathbf{d}_1|}$ and equipped with an equivalence class of graded coordinates. Here we assume that the number of even (resp. odd) coordinates of weight i is equal to $d_{0,i}$ (resp. $d_{1,i}$) where $1 \leq i \leq k$. Two systems of graded coordinates are *equivalent* if they are related by a polynomial transformation with coefficients in \mathbb{R} which preserve both the parity and the weights (cf. Definition 2.1).

A *morphism* between super graded spaces W_1 and W_2 is a map $\Phi : W_1 \rightarrow W_2$ which in some (and thus any) graded coordinates writes as a polynomial respecting the $\mathbb{N}_0 \times \mathbb{Z}_2$ -gradation.

Informally speaking, a super graded bundle is a collection of super graded spaces parametrized by a base supermanifold.

Definition 5.2. A *super graded bundle of rank \mathbf{d}* is a super fiber bundle $\pi : \mathcal{E} \rightarrow \mathcal{M}$ with the typical fiber $\mathbb{R}^{\mathbf{d}}$ considered as a super graded space of rank \mathbf{d} . In other words, there exists a cover $\{\mathcal{U}_i\}$ of the supermanifold \mathcal{M} such that the total space \mathcal{E} is obtained by gluing trivial super graded bundles $\mathcal{U}_i \times \mathbb{R}^{\mathbf{d}} \rightarrow \mathcal{U}_i$ by means of transformations $\phi_{ij} : \mathcal{U}_{ij} \times \mathbb{R}^{\mathbf{d}} \rightarrow \mathcal{U}_{ij} \times \mathbb{R}^{\mathbf{d}}$ of the form

$$\begin{aligned} [cc]y^{a'} &= \sum_{I,J} Q_{I,J}^{a'}(x, \theta) y^{a_1} \dots y^{a_i} \xi^{A_1} \dots \xi^{A_j}, \\ \xi^{A'} &= \sum_{I,J} Q_{I,J}^{A'}(x, \theta) y^{a_1} \dots y^{a_i} \xi^{A_1} \dots \xi^{A_j}. \end{aligned}$$

Here $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$; $Q_{I,J}^{a'}$ and $Q_{I,J}^{A'}$ are local functions of (super) coordinate functions (x^i, θ^α) on \mathcal{U}_{ij} ; (y^a, ξ^A) and $(y^{a'}, \xi^{A'})$ are graded super coordinates on fibers $\mathbb{R}^{\mathbf{d}}$; and the summation is over such sets of indices $I = (a_1, \dots, a_i)$ and $J = (A_1, \dots, A_j)$ that the parity and the weight of each monomial in the sums on the right coincides with the parity and the weight of the corresponding coordinate on the left.

The notion of a *morphism* $\Phi : \mathcal{E} \rightarrow \mathcal{E}'$ between super graded bundles is clear; it is enough to assume that Φ is a morphism of supermanifolds such that the corresponding algebra map $\Phi^* : \mathcal{O}_{\mathcal{E}'}(|\mathcal{E}'|) \rightarrow \mathcal{O}_{\mathcal{E}}(|\mathcal{E}|)$ preserves the \mathbb{N}_0 -gradation.

Example 5.3. Higher tangent bundles have their analogs in supergeometry. Given a supermanifold \mathcal{M} a higher tangent bundle $T^k \mathcal{M}$ is a natural example of a super graded bundle.⁴ For $k = 2$ and local coordinates (x^A) on \mathcal{M} (even or odd) one can introduce natural coordinates $(x^A, \dot{x}^B, \ddot{x}^C)$ on $T^2 \mathcal{M}$ where coordinates \dot{x}^A and \ddot{x}^A share the same parity as x^A and are of weight 1 and 2, respectively. Standard transformation rules apply:

$$x^{A'} = x^{A'}(x), \quad \dot{x}^{A'} = \dot{x}^B \frac{\partial x^{A'}}{\partial x^B}, \quad \ddot{x}^{A'} = \ddot{x}^B \frac{\partial x^{A'}}{\partial x^B} + \dot{x}^C \dot{x}^B \frac{\partial^2 x^{A'}}{\partial x^B \partial x^C}.$$

³In the context of supermanifolds we should rather speak about isomorphism than diffeomorphism. Since there is no concept of a topological supermanifold, there is no need to distinguish between homeomorphisms, C^1 -diffeomorphism or smooth diffeomorphisms.

⁴In a natural way higher tangent bundles correspond to the (purely even) Weil algebra $\mathbb{R}[\varepsilon]/\langle \varepsilon^{k+1} \rangle$ (see [KMS93]). In general, any super Weil algebra gives rise to a Weil functor that can be applied to a supermanifold (cf. [BF10]).

Homogeneity structures in the category of supermanifolds Also the notion of a homogeneity structure easily generalizes to the setting of supergeometry.

Definition 5.4. A *homogeneity structure* on a supermanifold \mathcal{M} is a smooth action $h : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}$ of the multiplicative monoid (\mathbb{R}, \cdot) of real numbers, i.e., h is a morphism of supermanifolds such that the following diagram

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathcal{M} & \xrightarrow{\text{id}_{\mathbb{R}} \times h} & \mathbb{R} \times \mathcal{M} \\ m \times \text{id}_{\mathcal{M}} \downarrow & & \downarrow h \\ \mathbb{R} \times \mathcal{M} & \xrightarrow{h} & \mathcal{M} \end{array}$$

commutes (here $m : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ denotes the standard multiplication) and that $h_1 := h|_{\{1\} \times \mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$ is the identity morphism. In other words, h is a morphism of supermanifolds defined by a collection of maps $h_t : \mathcal{M} \rightarrow \mathcal{M}$, $t \in \mathbb{R}$ such that $h_{ts}^* = h_t^* \circ h_s^*$ for any $t, s \in \mathbb{R}$ and that $h_1 = \text{id}_{\mathcal{M}}$.

A *morphism* of two homogeneity structures (\mathcal{M}_1, h_1) and (\mathcal{M}_2, h_2) is a morphism $\Phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ of supermanifolds intertwining the actions h_1 and h_2 . Clearly, homogeneity structures on supermanifolds with their morphism form a *category*.

Per analogy to the standard (real) case, we say that a local function $f \in \mathcal{O}_{\mathcal{M}}(U)$, where U is an open subset of $|\mathcal{M}|$, is called *homogeneous* of *weight* $w \in \mathbb{N}$ if

$$h_t^*(f) = t^w \cdot f,$$

for any $t \in \mathbb{R}$. We assume here that the carrier U of f is preserved by the action h , i.e. $h_t(U) \subset U$ for any $t \in \mathbb{R}$.

Remark 5.5. Observe that given a homogeneity structure h on a supermanifold \mathcal{M} the induced maps $h_t : |\mathcal{M}| \rightarrow |\mathcal{M}|$ equip the body $|\mathcal{M}|$ with a (standard) *homogeneity structure*, and so $h_0 : |\mathcal{M}| \rightarrow |\mathcal{M}|_0$ is a (real) *graded bundle* over $|\mathcal{M}|_0 := h_0(|\mathcal{M}|)$.

Note also that, analogously to the standard case, every super graded bundle structure $\pi : \mathcal{E} \rightarrow \mathcal{M}$ provides \mathcal{E} with a *canonical homogeneity structure* $h^{\mathcal{E}}$ defined locally in an obvious way. We call it an action of the *homotheties* of \mathcal{E} . Obviously, a morphism of super graded bundles $\Phi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ induces a morphism of the related homogeneity structures $(\mathcal{E}_1, h^{\mathcal{E}_1})$ and $(\mathcal{E}_2, h^{\mathcal{E}_2})$.

Alike in the standard case, homogeneous functions on super graded bundles are polynomial in graded coordinates.

Lemma 5.6. Let f be a homogeneous function on a trivial super graded bundle $\mathcal{U} \times \mathbb{R}^{\mathbf{d}}$, where \mathcal{U} is a superdomain and $\mathbf{d} = (\mathbf{d}_0 | \mathbf{d}_1)$ is as above. Then f is a homogeneous polynomial in graded fiber coordinates.

Proof. This follows directly from a corresponding result for purely even graded bundles. Indeed, let $f \in C^\infty(x, y_w^a)[\theta, \xi^A]$ be an even or odd, homogeneous function on $\mathcal{U} \times \mathbb{R}^{\mathbf{d}}$:

$$f = \sum_{I, J} f_{I, J}(x, y^a) \xi^{A_1} \dots \xi^{A_i} \theta^{B_1} \dots \theta^{B_j},$$

where (y^a, ξ^A) are graded coordinates on $\mathbb{R}^{\mathbf{d}}$, and the summation goes over sequences $I = \{A_1 < \dots < A_i\}$, $J = \{B_1 < \dots < B_j\}$. Then

$$h_t^* f = \sum_{I, J} f_{I, J}(x, t^{\mathbf{w}(a)} y^a) t^{\mathbf{w}(A_1) + \dots + \mathbf{w}(A_i)} \xi^{A_1} \dots \xi^{A_i} \theta^{B_1} \dots \theta^{B_j},$$

so $h_t^* f = t^w f$ implies that the coefficients $f_{I, J}$ are real functions of weight $w - (\mathbf{w}(A_1) + \dots + \mathbf{w}(A_i)) \geq 0$, thus polynomials in y^a . \square

In what follows we will need the following technical result which allows to construct homogeneous coordinates under mild technical conditions.

Lemma 5.7. *Consider a superdomain $\mathcal{M} = U \times \Pi\mathbb{R}^s$ with $U \subset \mathbb{R}^r$ being an open set and introduce super coordinates $(y^1, \dots, y^r, \xi^1, \dots, \xi^s)$ on \mathcal{M} , i.e. y 's are even and ξ 's are odd coordinates on \mathcal{M} . Let h be an action of the monoid (\mathbb{R}, \cdot) on \mathcal{M} such that*

$$h_t^*(y^a) = t^{\mathbf{w}(a)} y^a + o(\mathcal{J}_{\mathcal{M}}), \quad \text{and} \quad h_t^*(\xi^i) = t^{\mathbf{w}(i)} \xi^i + o(\mathcal{J}_{\mathcal{M}}^2).$$

Then

$$\left(\frac{1}{\mathbf{w}(a)!} \frac{d^{\mathbf{w}(a)}}{dt^{\mathbf{w}(a)}} \Big|_{t=0} h_t^*(y^a), \frac{1}{\mathbf{w}(i)!} \frac{d^{\mathbf{w}(i)}}{dt^{\mathbf{w}(i)}} \Big|_{t=0} h_t^*(\xi^i) \right)$$

are graded coordinates on the superdomain \mathcal{M} .

Proof. We remark that if

$$h_t^* f = \sum_{I \subset \{1, \dots, s\}} g_I(t, y^1, \dots, y^r) \xi^I \in \mathcal{C}^\infty(\mathbb{R} \times U)[\xi^1, \dots, \xi^s]$$

is a function on \mathcal{M} , then the function $f^{[k]} := \frac{1}{k!} \frac{d^k}{dt^k} \Big|_{t=0} h_t^* f$ is well-defined as h is smooth and is given by

$$f^{[k]} = \frac{1}{k!} \sum_{I \subset \{1, \dots, s\}} \xi^I \frac{d^k}{dt^k} \Big|_{t=0} g_I(t, y^1, \dots, y^r) \in \mathcal{C}^\infty(U)[\xi^1, \dots, \xi^s].$$

Now, since for any morphism $\phi : \mathcal{M} \rightarrow \mathcal{M}$, $(\text{id}_{\mathbb{R}} \times \phi)^* : \mathcal{O}_{\mathbb{R} \times \mathcal{M}} \rightarrow \mathcal{O}_{\mathbb{R} \times \mathcal{M}}$ commutes with the operators $\frac{d^k}{dt^k} \Big|_{t=0} : \mathcal{O}_{\mathbb{R} \times \mathcal{M}} \rightarrow \mathcal{O}_{\mathbb{R} \times \mathcal{M}}$, we get

$$h_s^* f^{[k]} = \frac{1}{k!} \frac{d^k}{dt^k} \Big|_{t=0} h_s^* h_t^* f = \frac{1}{k!} \frac{d^k}{dt^k} \Big|_{t=0} h_{st}^* f = s^k \frac{d^k}{dt^k} \Big|_{t=0} h_t^* f = s^k f^{[k]},$$

that is, $f^{[k]}$ is h -homogeneous of weight k . In particular,

$$(y^a)^{[\mathbf{w}(a)]} = y^a + o(\mathcal{J}_{\mathcal{M}}) \quad \text{and} \quad (\xi^i)^{[\mathbf{w}(i)]} = \xi^i + o(\mathcal{J}_{\mathcal{M}}^2),$$

are homogeneous with respect to h . To prove that these are true coordinates on \mathcal{M} observe that the matrices $\left(\frac{\partial (y^a)^{[\mathbf{w}(a)]}}{\partial y^b} \right)$ and $\left(\frac{\partial (\xi^i)^{[\mathbf{w}(i)]}}{\partial \xi^j} \right)$ are invertible, so the result follows. \square

The main result We are now ready to prove that Theorem 2.9 generalizes to the supergeometric context.

Theorem 5.8. *The categories of super graded bundles (with connected bodies) and homogeneity structures on supermanifolds (with connected bodies) are equivalent. At the level of objects this equivalence is provided by the following two constructions*

- With every super graded bundle $\pi : \mathcal{E} \rightarrow \mathcal{M}$ one can associate a homogeneity structure $(\mathcal{E}, h^{\mathcal{E}})$, where $h^{\mathcal{E}}$ is the action by the homotheties of \mathcal{E} .
- Given a homogeneity structure (\mathcal{M}, h) on a supermanifold \mathcal{M} , the map $h_0 : \mathcal{M} \rightarrow \mathcal{M}_0 := h_0(\mathcal{M})$ provides \mathcal{M} with a canonical structure of a super graded bundle such that h is the related action by homotheties.

At the level of morphisms: every morphism of super graded bundles is a morphism of the related homogeneity structures and, conversely, every morphism of homogeneity structures on supermanifolds respects the canonical super graded bundle structures.

Proof. The crucial part of the proof is to show that given a homogeneity structure $h : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}$ on a supermanifold \mathcal{M} one can always find an atlas with homogeneous coordinates on \mathcal{M} . First we observe that without loss of generality we may assume that \mathcal{M} has a simple form, namely \mathcal{M} is isomorphic with $U \times \mathbb{R}^{\mathbf{d}} \times \Pi \mathbb{R}^q$ for some small open subset $U \subset \mathbb{R}^n$, i.e. \mathcal{M} has a second, other than h , homogeneity structure associated with a vector bundle $E = U \times \mathbb{R}^{\mathbf{d}} \times \mathbb{R}^q \rightarrow U \times \mathbb{R}^{\mathbf{d}}$. Using the fact that these graded bundle structures are compatible, and transferring the homogeneity structure h to the real manifold E (with some loss of information) we are able to construct graded coordinates for \mathcal{M} but modulo $\mathcal{J}_{\mathcal{M}}^2$. Then we evoke Lemma 5.7 to finish the proof.

Assume that $h : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}$ is a homogeneity structure on a supermanifold \mathcal{M} . Recall (see Remark 5.5) that h induces a canonical homogeneity structure \underline{h} on the body $|\mathcal{M}|$. Since we work locally we may assume without any loss of generality that $|\mathcal{M}|_0 := \underline{h}_0(|\mathcal{M}|)$ is an open contractible subset $U \subset \mathbb{R}^n$, and $|\mathcal{M}| = U \times \mathbb{R}^{\mathbf{d}}$ is a trivial graded bundle over U of rank $\mathbf{d} = (d_1, \dots, d_k)$. Thus we may assume that $\mathcal{M} = \Pi E$ where $E = U \times \mathbb{R}^{\mathbf{d}} \times \mathbb{R}^q$ is a trivial vector bundle over $|\mathcal{M}| = U \times \mathbb{R}^{\mathbf{d}}$ with the typical fiber \mathbb{R}^q . Note, that we do not need to refer to Batchelor's theorem [Gaw77, Bat79] and the argument works even for holomorphic actions of the monoid (\mathbb{C}, \cdot) on complex supermanifolds (see Remark 5.9).

Consider now local coordinates (x^i, y_w^a, Y^A) on E where (x^i, y_w^a) are graded coordinates on the base and Y^A are linear coordinates on fibers. Let (ξ^A) be odd coordinates on ΠE corresponding to (Y^A) . Recall that $\mathcal{J}_{\mathcal{M}}(|\mathcal{M}|) = \langle \xi^A \rangle$ denotes the nilpotent radical of $\mathcal{O}_{\mathcal{M}}(|\mathcal{M}|)$. Since (x^i, y_w^a) are graded coordinates with respect to \underline{h} and since h_t respects the parity for each $t \in \mathbb{R}$, the general form of h_t must be

$$(30) \quad \begin{cases} h_t^*(x^i) &= x^i + o(\mathcal{J}_{\mathcal{M}}^2), \\ h_t^*(y_w^a) &= t^w y_w^a + o(\mathcal{J}_{\mathcal{M}}^2), \\ h_t^*(\xi^A) &= \alpha_B^A(t, x^i, y_w^a) \xi^B + o(\mathcal{J}_{\mathcal{M}}^2), \end{cases}$$

where α_B^A are smooth functions.

The action h defines an action \tilde{h} of the monoid (\mathbb{R}, \cdot) on E which is given by

$$(31) \quad \tilde{h}_t^*(x^i) = x^i, \quad \tilde{h}_t^*(y^a) = t^{w(y^a)} y^a, \quad \text{and} \quad \tilde{h}_t^*(Y^A) = \alpha_B^A(t, x, y) Y^B.$$

Indeed, by reducing $h_t^* : \mathcal{O}_{\mathcal{M}}(|\mathcal{M}|) \rightarrow \mathcal{O}_{\mathcal{M}}(|\mathcal{M}|)$ modulo $\mathcal{J}_{\mathcal{M}}^2(|\mathcal{M}|)$ we obtain an endomorphism of $\mathcal{O}_{\mathcal{M}}(|\mathcal{M}|)/\mathcal{J}_{\mathcal{M}}^2(|\mathcal{M}|) = \mathcal{C}^\infty(x^i, y_w^a) \oplus \xi^A \cdot \mathcal{C}^\infty(x^i, y_w^a) \simeq \mathcal{C}^\infty(x^i, y_w^a) \oplus Y^A \cdot \mathcal{C}^\infty(x^i, y_w^a)$, thus $\tilde{h}_t \circ \tilde{h}_s = \tilde{h}_{ts}$ and \tilde{h}_t does not depend on a particular choice of linear coordinates Y^A on E . It follows from Theorem 2.9 that E is a graded bundle over $E_0 := \tilde{h}_0(E)$, whose homotheties coincide with the maps \tilde{h}_t . Note that the inclusions $U \times \{0\} \times \{0\} \subset E_0 \subset U \times \{0\} \times \mathbb{R}^q$ can be proper.

Our goal now is to find graded coordinates on E out of non-homogeneous coordinates (x^i, y^a, Y^A) and then mimic the same changes of coordinates in order to define a graded coordinate system on the supermanifold $\mathcal{M} = \Pi E$ out of a non-homogeneous one (x^i, y^a, ξ^A) .

Denote by H the homotheties related with the vector bundle structure on $\tau : E \rightarrow |\mathcal{M}| = U \times \mathbb{R}^{\mathbf{d}}$. A fundamental observation that follows from (31) is that the actions H and \tilde{h} commute, i.e.,

$$H_s \circ \tilde{h}_t = \tilde{h}_t \circ H_s$$

for every $t, s \in \mathbb{R}$. Thus (E, \tilde{h}, H) is a *double homogeneity structure* and, by Theorem 5.1 of [GR11], a

double graded bundle:

$$\begin{array}{ccc} E = U \times \mathbb{R}^d \times \mathbb{R}^q & \xrightarrow{H_0} & U \times \mathbb{R}^d \\ \downarrow \tilde{h}_0 & & \downarrow \tilde{h}_0|_{U \times \mathbb{R}^d} \\ E_0 & \xrightarrow{H_0|_{E_0}} & U. \end{array}$$

Moreover, the above-mentioned result implies that we can complete graded coordinates (x^i, y_w^a) on $U \times \mathbb{R}^d$ which are constant along fibers of the projection H_0 with graded coordinates \tilde{Y}_w^A of bi-weight $(w, 1)$, where $0 \leq w \leq k$ so that $(x^i, y_w^a, \tilde{Y}_w^A)$ is a system of bi-graded coordinates for (E, \tilde{h}, H) .

Since both (Y^A) and (\tilde{Y}_w^A) are linear coordinates for the vector bundle $H_0 : E \rightarrow U \times \mathbb{R}^d$ they are related by

$$(32) \quad \tilde{Y}_w^A = \gamma_B^A(x, y) Y^B$$

for some functions γ_B^A on $U \times \mathbb{R}^d$.

Let us define $\tilde{\xi}^A := \gamma_B^A(x, y) \cdot \xi^B$, i.e. using the same functions as in (32). By applying \tilde{h}_t^* to (32) we get

$$t^{\mathbf{w}(A)} \gamma_C^A(x, y) Y^C \stackrel{(32)}{=} t^{\mathbf{w}(A)} \tilde{Y}_w^A = \tilde{h}_t^*(\tilde{Y}_w^A) \stackrel{(32)}{=} \tilde{h}_t^*(\gamma_B^A(x, y)) \tilde{h}_t^*(Y^B) \stackrel{(31)}{=} \gamma_B^A(x, t^w y_w^a) \alpha_C^B(t, x, y) Y^C,$$

hence $t^{\mathbf{w}(A)} \gamma_C^A = \gamma_B^A(x, t^w y_w^a) \alpha_C^B(t, x, y)$, and

$$\begin{aligned} h_t^*(\tilde{\xi}^A) &= h_t^*(\gamma_B^A) h_t^*(\xi^B) \stackrel{(30)}{=} \gamma_B^A(x, t^w y_w^a + o(\mathcal{J}_{\mathcal{M}}^2)) (\alpha_C^B(t, x, y) \xi^C + o(\mathcal{J}_{\mathcal{M}}^2)) = \\ (33) \quad &= \gamma_B^A(x, t^w y_w^a) \alpha_C^B(t, x, y) \xi^C + o(\mathcal{J}_{\mathcal{M}}^2) = t^{\mathbf{w}(A)} \gamma_C^A(x, y) \xi^C + o(\mathcal{J}_{\mathcal{M}}^2) = \\ &= t^{\mathbf{w}(A)} \tilde{\xi}^A + o(\mathcal{J}_{\mathcal{M}}^2). \end{aligned}$$

We obtain a graded coordinate system for \mathcal{M} due to Lemma 5.7.

The equivalence at the level of morphism follows directly from Lemma 5.6. This result implies that locally any supermanifold morphism respecting the homogeneity structures is a homogeneous polynomial in graded coordinates, i.e. it is a morphism of the related super graded bundles (cf. Definition 5.2). \square

Remark 5.9. Using the same methods one can prove an analog of above result for holomorphic supermanifolds (a super-version of complex manifolds): a holomorphic action of (\mathbb{C}, \cdot) on a holomorphic supermanifold \mathcal{M} gives rise to a graded holomorphic super coordinate system for \mathcal{M} . Indeed, the proof of Lemma 5.7 can be rewritten in a holomorphic setting. The other result we need to complete the proof of Theorem 5.8 in the holomorphic context is that two holomorphic commuting (\mathbb{C}, \cdot) actions on a complex manifold M give rise to $\mathbb{N}_0 \times \mathbb{N}_0$ graded coordinate system on M (an analog of Theorem 5.1 [GR11]). This can be justified using a double graded version of Lemma 3.11 and the fact that M can be considered as a substructure of $J^r J^s M$ for some r, s (a double holomorphic homogeneity substructure). Details are left to the Reader.

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